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MATHEMATICAL EDUCATION IN THE AMERICAS

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A Report of the First Inter-American Conference on Mathematical Education

Bogotá, Colombia, December 4 to 9, 1961

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FOREWORD

Under the sponsorship of the International Commission on Mathematical Instruction[1] and the Organization of American States, the first Inter-American Conference on Mathematical Education was held at Bogotá, Colombia, from December 4 to 9, 1961. Representatives and invited guests from twenty-three countries of the Western Hemisphere convened to explore methods of improving mathematics education at the secondary and university levels through a program of inter-American cooperation. The deliberations and conclusions of this conference are reported in this volume and in its Spanish-language counterpart.

To achieve the goals of the Conference, mathematicians and mathematics educators from American countries, as well as a small number of outstanding mathematicians from Europe, were invited to present addresses on contemporary mathematics and on problems in the teaching of the subject. Formal discussion periods were scheduled to follow all addresses which, from the nature of their titles, promised to elicit lively debate and questions. The addresses, together with digests of remarks made during the discussion periods, are included in this report.

Since one of the items on the agenda of the Conference was the adoption of a set of resolutions embodying suggestions and plans for future cooperation, surveys of the present situation in mathematical education in each of the countries of the Americas were prepared and distributed among the delegates. A brief summary of this survey is also included in this report.

Both the Conference and the publication of the English and Spanish editions of the report were made possible by grants from a number of philanthropic and scientific organizations. The debt of the Organizing Committee of the Conference to these groups for their generous support is duly acknowledged in Part III. The preparation of the report was under the direction of Professor Howard F. Fehr (U.S.A.), Secretary of the Conference, assisted by Dr. Linda Allegri (U.S.A.).

[1] The Commission, of which Professor Marshall H. Stone (U.S.A.) is President, was created by the International Mathematics Union to consider problems concerning mathematical education.

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MATHEMATICAL
EDUCATION
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AMERICAS

Part I

OPENING ADDRESSES

*WELCOMING ADDRESS**

Dr. Jaime Posada, Minister of Education, Colombia

In America, we are building a whole new way of life and culture. There are many manifestations of what we are doing which clearly delineate a new world and epoch.

In connection with established practices and procedures of international relationships, our countries and our people have been able to evolve a regimen of solidarity and cooperation which is the most auspicious of this era. The instruments of government, modes of action, collective laws and statutes that characterize, maintain, and perfect it set an example for the world.

The Act of Bogotá and the Conference of Punta del Este in turn proposed a set of postulates and instruments most likely to give impetus to economic growth and social development, and to demonstrate to the needy of the hemisphere that inter-Americanism is not a fruitless dream but—and this is of utmost importance—the determination to give to the people a better and more progressive frame of security within which to exercise its dignity.

Those of us who are assembling here today, however, know that all would be of no avail, and would seem truncated without the stirring presence of a strong intellectual program, without the improvement of education at all levels, without the expansion and transformation of the services of the universities, without inducements to carry on research, to write, to teach, and, indeed, without a foundation sufficiently serious and active to establish harmony and reciprocity among those of our men who, like you, are concerned with science,

*Address translated from the Spanish.

believe in it, encourage its study in the world, and develop it into a prodigious and decisive phenomenon of the creative mind.

Colombia, which has been fortunate in welcoming visitors who belong to different groups, acknowledges and celebrates in the manner it deserves this opportunity which you, your organizations, and your countries have accorded her as the place where a great, new, and attractive enterprise for the welfare of mathematics is to be initiated. On behalf of the Governor, the Ministry of Education, the Association of Universities, the Local Organizing Committee of the Conference, I give testimony of the satisfaction and gratitude with which this honor was received, and assure you that you are among people of good will who are pleased with your presence here.

In particular, I wish to stress the unfailing cooperation that has been given to the success of this effort by the Department of Scientific Development of the Organization of American States, UNESCO, the Ford Foundation, the Rockefeller Foundation, the National Science Foundation (U.S.A.), the International Organizing Committee, the Association of Universities of Colombia, and also the Local Organizing Committee.

The group of invited speakers, Professors Schwartz, Choquet, Stone, Fehr, Bundgaard, Pauli, Cansado, and Torres, and the eminence of the participants and observers representing twenty-three nations guarantee the seriousness of the labors of the Conference and the implementation of its conclusions. In view of the intentions we expressed and the proposals we made when we promoted and sponsored the plans for this meeting a year ago, we cannot be satisfied with a pleasant but transitory conference without future action.

The inter-American union must continue to expand its cultural, scientific, and university spheres. It would be logical to continue to hold conferences of this type periodically and regularly. With the sponsorship of the Department of Scientific Development of the Pan American Union, it would be advisable to create permanent commissions on mathematics, on physics, on chemistry, on biology, just as it was possible to create an efficient and well-functioning commission on nuclear affairs.

Commissions of this type would permit us, on both a continental and a national scale, to change or emphasize the changes in methods of teaching science, to utilize the most recent and most appropriate textbooks and types of experimental laboratories, and to keep abreast of new developments. If, moreover, as has been recommended lately by committees of experts and has been adopted by the Conference of Punta del Este, the university and educational services of the Pan American Union were strengthened, and an Inter-American University Fund were created, as was proposed by Colombia, we would widen the perspective of a vast movement of coordination and reformation.

This is a congress of ideas—representative of the most serious and, therefore, most responsible and constructive thinking in America. Let us consequently take cognizance, with satisfaction, of the days of comparison and clarification that it will offer us.

RESPONSE TO WELCOMING ADDRESS OF DR. JAIME POSADA

Professor Marshall H. Stone (U.S.A.)

It is a great privilege to respond to our generous hosts and to the distinguished Minister who has spoken for them here this evening. In doing so, I act in three capacities: in the name of the International Commission on Mathematical Instruction, which has proposed and organized this first Inter-American Conference on Mathematical Education; in the name of the Organizing Committee appointed by the Commission to realize the Conference; and in the name of all who are attending the Conference.

It is deeply satisfying to note that this conference, which will deal with such difficult and fundamental problems as those confronting us in mathematical education, has received so warm and friendly a welcome in this beautiful land situated in the heart of the Americas. It is fitting that our deliberations are taking place in a city long dedicated to the things of the intellect and of the spirit, a city which proudly maintains its tradition as the Athens of the Americas.

The International Commission on Mathematical Instruction and the Organizing Committee are heavily in debt to the committee created by the Colombian Association of Universities to plan all the complicated details which must be executed on the spot if such a conference as ours is to proceed smoothly and agreeably for all concerned. On behalf of the Organizing Committee, it is a pleasure to bear witness to the enthusiastic interest displayed by this local committee and to thank all its members, past and present, most warmly for their generous contribution to the success of our meeting. Perhaps I may be allowed for just a moment to speak also for the sponsors of the Conference, adding their thanks to those of the Organizing Committee for the very valuable work of the local committee under the leadership first of Dr. Bejarano, and presently of Dr. Zabala. These sponsors, as everyone is informed, comprise, in alphabetical order, the following institutions: the Colombian Association of Universities, the Ford Foundation, the National Science Foundation (U.S.A.), the Organization of American States, the Rockefeller Foundation, and UNESCO.

Several of our sponsors, in addition to giving us continual encouragement and extremely generous material support in our purpose of coming to grips with urgent contemporary problems in the teaching of mathematics, have sent observers to sit with us during the coming days. I take this opportunity to welcome them individually: Dr. Marcelo Alonso, the delegate of the O.A.S.; Dr. Oscar Dodera Luscher, the delegate of UNESCO; and Dr. Bowen C. Dees, the delegate of the National Science Foundation (U.S.A.). At the same time, I welcome all others who are here in the capacity of observers by whomever they may have been sent. And to the sponsoring organizations actually represented here, and to those present only in spirit, I express our most sincere thanks for enabling us to meet under such highly favorable conditions for the achievement of a lofty aim.

This aim is nothing less than to do our utmost here toward bringing mathematical education in the several countries of which we are citizens up to the level of our times—times which demand that our young people shall be better mathematicians than their fathers, far more imaginative, and far more skillful in making mathematics fruitful in the varied affairs of mankind. This is not the moment to dissert upon the tasks we are about to take up in our working sessions, nor am I the person indicated to pronounce a discourse concerning them in this gathering. Nevertheless, this distinguished audience may perhaps kindly permit me to observe that its presence here in Bogotá is the strongest possible evidence of the widespread interest and concern aroused by the current problems in mathematical education—and the best possible guarantee of our determination to understand and solve them. If there be any need to underscore the urgent and practical nature of the task we have set ourselves, I may recall a few salient facts, more or less unrelated to one another, which can only strengthen the convictions each of you has brought here on the basis of his own knowledge and experience.

First of all, let me cite the marked increase in the demand for mathematical training which has been noted in many different quarters over the last few years. In a great many universities, enrollments in mathematics courses have shot up—in some cases they have even tripled—with the result that mathematics departments are overtaxed, if not overwhelmed. Efforts to meet this demand by the provision of new facilities, expanded staffs, and more elaborately organized departments or institutes have too often fallen short of their mark. Indeed, it is not infrequent that a projected mathematical institute is recognized to be inadequate or even obsolescent before its doors have been opened to mathematics students and research workers.

To some extent this prominent student interest reflects economic facts which have lately become the object of searching analysis by such pragmatically oriented bodies as the Organization for European Economic Cooperation, now known as the Organization for Economic Cooperation and Development. These studies agree in focusing attention on the central role played in economic development by scientific research and by the improvement of mathematical, scientific, and technical education. Even in the case of the most advanced industrial countries, one of the most vital contributions to economic progress, the O.E.C.D. has concluded, lies in the improvement of science and mathematics teaching. Not content with giving publicity to this conclusion, O.E.C.D. has undertaken to initiate educational reforms. In order to define the shape which such improvements should take in the field of mathematics teaching, O.E.C.D. called a conference somewhat similar to this one, at Royaumont, near Paris, late in 1959. It went on to sponsor the elaboration of detailed proposals for new secondary school mathematics curricula, now embodied in the so-called Dubrovnik Report, under the title, "Synopses for Modern Secondary School Mathematics." At present it is engaged in fostering experimentation by certain European schools with courses based on or inspired by this report.

In the United States a far more intensive and elaborate effort is being made toward similar ends, though under basic conditions markedly different from those obtaining in Europe. In the United States, the improvement of the mathematical curriculum is inevitably involved in a concurrent drive to raise the standards of student and teacher performance, whereas in Europe, reform has to be coupled with a vigorous attempt to draw a larger proportion of the coming generations into the secondary schools and universities. We are very fortunate indeed in having as a participant in this conference the leader of the principal organization carrying on this important work in the United States, Professor E. G. Begle, Director of the School Mathematics Study Group, now located at Stanford University.

While we are accustomed to think of mathematics as a subject which is vital for the study of physics, chemistry, and engineering—even for some branches of biology, such as genetics and population theory—we now have to broaden our perspective to encompass many new and unexpected applications of mathematics to the social sciences as well as to hitherto unaffected parts of biology. In the United States, for example, many economists are now prepared to take the position that fairly extensive mathematical training must be required of all students of economics. However, the most decisive indication of such current trends is found in the recent action of the Facultés de Droit et des Sciences Économiques in France, instituting a separate four-year degree program in economics and making mathematics and statistics compulsory subjects during the first three years.

These are, of course, only a few scattered instances of the changes and the new demands which are conspiring to bring about an important modification of the relation of mathematics to the entire educational scheme. They are, quite plainly, something more than mere straws in the wind. I am sure that you can add to these many others from your own experience, and that you will all agree on the need for giving most careful consideration at this conference to their implications for the teaching of mathematics in the countries of America.

After these somewhat disjointed and necessarily brief remarks upon the nature and the importance of our task, it remains for me to welcome in the name of the International Commission on Mathematical Instruction all those who are here as participants, speakers, or observers. We are particularly grateful to the four speakers who have consented to come here from the other hemisphere—Professors Bundgaard, Choquet, Pauli, and Schwartz. We wish also to express our appreciation of the interest which certain of the American governments have shown in our work by sending observers as their official representatives at these meetings.

The Commission is deeply gratified by this interest in the cause to which it is dedicated. If, as the President of the Commission, I were to express any hope beyond the natural one that this may be a supremely successful conference, it would be that many American countries in addition to the Argentine Republic and the United States might be inspired to associate themselves permanently with the work of the Commission after this conference has adjourned. The problems of mathematical education are universal in character despite variations and nuances produced by local conditions. Their solution demands the fullest cooperation of mathematicians and mathematics teachers on an international level. As an organ of the International Mathematical Union, the Commission aims to foster and to encourage such cooperation in every possible way, and is deeply desirous of placing its work upon a still broader basis than that already provided by the active support of twenty-two different nations located on five of the six continents.

Part II

ADDRESSES BY THE INVITED SPEAKERS

Because of their importance in directing the thinking of the Conference and in evoking discussion which culminated in the drafting of a set of resolutions for future action, the addresses of the invited speakers are reproduced, for the most part, in their entirety and in the order in which they were presented. Statements made during the discussion periods that followed certain addresses are presented in summary form.

*MATHEMATICS AND OUR TECHNOLOGICAL SOCIETY**

Professor Alberto González Dominguez (Argentina)

Allow me, colleagues, to begin my exposition with a prophetic paragraph from the writings of the Father of Modern Science—words which he puts in the mouth of "Il Saggiatore":

> "Signor Sarsi, it is not like that. Philosophy is written in this great book which is constantly open in front of your eyes (I mean the universe), but it cannot be understood if you don't first learn the language and know the alphabet in which it is written. It is written in the language of mathematics and the alphabet is composed of triangles, circles, and other geometric figures, without which tools, it is humanly impossible to understand a single word, and without these, it would be a matter of circling in vain inside a dark labyrinth."[1]

It is the destiny of genius to exceed its own limits; and the words of "Il Saggiatore," under whose protection I have placed this

*Address translated from the Spanish.

[1] Translated from the Italian. Galileo Galilei, Il Saggiatore, Opere, Vol. I, ed. Adriano Salani, Florence, Italy: A. Salani, 1935, pp. 147-148.

conference, sound in our ears with a strange ring of modernism three centuries after they were pronounced.

Technology is the child of science. Modern science—empirical Pythagoreanism—cannot be separated from its language, which is mathematics. Therefore, and this is the point of departure from which I start my discourse, mathematics is at the very base of our technological development.

It is my intention, in this address, to mention, even if very briefly (since neither my abilities nor the time element would permit me to do more), some of the more notable relations between mathematics and technology. The conclusion to which we will arrive is that, as a product of those interactions, on one hand, and, on the other hand, the extraordinary process which is actually in the course of unifying and formalizing all of mathematics, it is now possible to have a glimpse of the dawn of a new golden age of mathematics.

"Philosophy is written in the language of mathematics."

It is my plan to present a summary, which necessarily will be fragmentary and incomplete, of examples in which mathematico-technological interactions are more obvious. To do this with some order, I have found it convenient to refer to what are perhaps the most spectacular events of our technological age: the advent of controlled nuclear energy on one hand, and, on the other hand, the development of automatization to an unprecedented degree.

Let us begin.

We will have entered completely into the technological era when the energy at our disposal is as plentiful and inexpensive as the light of the sun and the air we breathe; and it has been clear since the day not so long ago when Fermi started his battery in Chicago that this desideratum is not a utopia but a certainty whose realization is only a matter of time. The extraordinary development of the technology of nuclear fission since that event supports this view. There is no doubt that this development is only a modest beginning which has not yet been able to modify the supply of energy or its price.

The decisive step will probably require the abandonment of fission as a basic method for obtaining energy and its replacement partially or totally by more "natural" nuclear reactions, fusion among them. This will culminate in the production of a motor in series or a nuclear generator, with great yield, which will transform nuclear energy directly into electricity.

In the same manner in which our children laugh with glee and disbelief when, on a rare occasion (becoming more rare, alas, every day), they see one of those old Fords which were our delight in the twenties, their own children will smile when, during a school visit to a museum, their teacher will explain that the incredible object before their eyes is a reactor of the sixties.

"But what about mathematics?" the impatient listener asks. This question brings me back to the topic.

Mathematics has more bearing on the problem than many of my listeners might suspect. It is my thesis, and it is easy to prove, that if we do not yet have the motor I just described, it is because we have met, among other things, mathematical difficulties along the way.

Someone said that the most practical thing in the world is a good theory, and this applies in our case to a great extent. The chances of anyone constructing an efficient nuclear motor without basing his work on a good theory of nuclear forces are very slim indeed. How can one think that Marconi preceded Maxwell and Hertz, or that von Siemens preceded Faraday and Ampère?

To be more precise: modern nuclear technology lacks a sufficiently rigorous and defined theory of nuclear forces with which to support itself. And at this point, mathematics finally makes its appearance.

Let me explain what I mean. Classical quantum mechanics, a marvelous creation of that happy period for physics which covers the years 1924-1932, and which acquired definitive mathematical form in the famous book by von Neumann, is a rigorous theory (i.e., mathematical) that accounts satisfactorily for phenomena which concern orbital electrons. Quantum mechanics not only has led to the mathematization, to an unprecedented degree, of molecular theory and theoretical chemistry, but has also been the indispensable forerunner of the later developments which led to the fission of uranium and, in the end, to nuclear technology.

However, the reaction to an opposing point of view is no less interesting. For in effect, the most important lesson to be derived from von Neumann's book consists in showing in an irrefutable manner that the theory of Hilbert space and its linear operators (i.e., functional analysis) is the natural mathematical tool of quantum mechanics. This was the first successful application of abstract mathematics in applied mathematics. Functional analysis suddenly became the major preoccupation of some of the better mathematical minds and the very fertile influence of this movement exists even at the present time, branching off in many directions.

But let us not be too quick to declare victory. That there are difficulties, not, however, unsurmountable, obstructing the transition from the mechanics of Newton (having a finite number of degrees of freedom) to the theory of elasticity and hydrodynamics (having an infinite number of degrees of freedom) is well known.

Moreover, it is a fact (which we do not dare to call unfortunate since a world with as much free energy per person as we could imagine, but with no interesting mathematical problems, would be intolerable)—it is a fact, I repeat, that similar difficulties exist also in quantum mechanics, but increased to an extraordinary extent.

Even the most simple case of quantum electrodynamics where only two classes of particles intervene or interact, photons (or light) and electrons (or matter), which in addition are very much better known and studied, offers mathematical difficulties which up to now have not been resolved. If my listeners realize that the number of particles and antiparticles can now be counted by the dozen, and that every week the newspapers let us know of the birth of some new one of those creatures called cosmotrons which could more aptly be called "destroyers of theories"—they will agree with me when I say that the elaboration of a rigorous theory of fundamental particles must of necessity offer formidable mathematical difficulties. Such a theory is, nevertheless, the only thing that would permit technology to advance decisively.

It would be advantageous to stop and inquire where those difficulties lie, and in doing this I shall refer to the simple case of electrodynamics.

I said a little while ago that quantum electrodynamics is nothing more than the theory of the interactions of electrons and photons, and the system of differential equations in which the theory finds mathematical expression is nothing but the classical equations of Maxwell (for the electromagnetic field) together with the equations of Dirac, which are the equations of the electron. The unknowns are precisely $A(x)$, electromagnetic potential, and $\psi(x)$, the spinor function which describes the electronic field.

The equations of Maxwell and the equations of Dirac have the agreeable property of being linear; the Maxwell-Dirac equations, on the other hand, are not linear, since they have a term which translates, in a precise manner, the interaction between the two fields, and in which the two unknowns $A(x)$, $\psi(x)$ appear to have been multiplied.

And here lies the disconcerting feature of the whole problem. If the equations are integrated, using, for want of a better method, the very much applied method of development in series, one finds to his astonishment that the series not only diverges, but its terms are all infinite starting from the second term on. Here is where the famous divergencies make their appearance—divergencies which are a fearful plague infecting the whole theory and which up to now have not been stamped out.

Where does the mistake lie? The explanation involves the fact that the modest nonlinear addend $A\psi$, seemingly harmless, in reality not only is not harmless but produces fatal results, since A and ψ are not functions but distributions and distributions generally cannot be multiplied. It is not my intention to put the blame for these divergencies on the talented inventor of distributions. On the contrary, I propose to show that the crucial problems in the theory of nuclear forces, and therefore for nuclear technology, are actually stated in terms of distributions, and certainly not simple scalar distributions, but vector distributions, which assume values in a space whose points in turn are operators in a Hilbert space.

In reality, the most elaborate mathematical theories enter into the theory of nuclear forces and into quantum field theory. The following are some of them: theory of groups (and, in particular, the Lorentz group, which is not compact), measure, and integrals in abstract spaces (the famous "integral along a path," remarkable invention of Riemann, is essentially an integral defined in a Hilbert space); tensor products in a Hilbert space; the famous theorem of Schwartz on the nucleus; analytic continuation of functions of several complex variables; multidimensional singular integrals (which provide the tools for the "relations of dispersion"); generalized theory of eigenfunctions and eigenvalues and many other theories and topics from the highest levels of contemporary mathematics. I refer my hearers to the extraordinary and magnificent book by Schwartz on Quantum Mathematics and Physics, for verification of the fact that I have not mentioned all of them.

Time, however, is running out and I must get to the topic of the interactions of mathematics and physics which group themselves around what I have called "automatization."

Although it is a curious phenomenon, it is nevertheless true that games, toys, and robots have held a fascination for philosophers and mathematicians from ancient times. The robots of philosophers, however, existed only in books. Recall Bacon and his island inhabited by sages who were treated like kings by marvelous machines; or the "man-machine" of La Mettrie, a frivolous book with nothing to recommend it aside from the title. Mathematicians have often been toy makers. Newton, while still a child, made his legendary "motor run on mouse-blood"; Lewis Carroll fascinated his little models by showing them marvelous collections of toys he had made; Chebyshev spent all his money on materials to build his talking mechanisms, his toys, and his computing machines. This was not time wasted for science, since the so-called "theory of approximations" of Chebyshev (or minimax approximation), which is today a brilliant chapter in the theory of functions, with many varied applications to optimum circuits called "filters," was a definite result of his desire to construct an "optimum" approximation for one of his talking mechanisms.

Let us recall the "Turing machine," which without even having emerged from the head of the author has revolutionized modern logic and, on the side, has influenced--and will influence even more in the future—the effective design of the most complex robot, which is the modern electronic computer. Last but not least, recall the universal von Neumann. I believe I am correct in saying that the first report by von Neumann in 1945 was the basis for the construction of the "DVAC" and that the second in 1947 (written in collaboration with Burks and Goldstine) contained a design of a computer capable of making 20,000 operations a second, with both dates marking decisive points in the history of this development; perhaps the most original and representative of the technological age. I wish to emphasize that it is not a matter of chance that the two great theoreticians of automatization—in certain aspects different but complementary—have been a mathematician who possessed a universal mind and a logician who had a mind that was the most profound and original of our epoch.

I will return to this topic later. Now, however, I wish to review briefly the development of the technological aspects of automation.

The robots and toys of mathematicians to which I have referred have exerted no influence on technology. The governor of Watt, on the contrary, had a distinctive role when it was incorporated into his steam engine. It is not an exaggeration to state that the industrial revolution was born under the sign of the automatic control. I will skip the distribution of controls invented after the governor of Watt, to come to the triode, one of the capital inventions of the twentieth century. The marvelous "valve without inertia" of De Forest marks the decisive step in the technology of automatic control. Another feature, almost as important, consists of the transistor, which was the collective invention of the physicists of the Bell Telephone Laboratories.

Modern electronic computers, which would not have been possible without the triode, use these parts by the thousands, and their speed reaches a million operations a minute. They have permitted the use of computers to obtain results which would otherwise have taken scores of years of work.

What I have just said is especially true for problems whose solutions depend upon the integration of functional equations and particularly of differential equations (if they are not linear). This brings me to the classical field of mathematics which has dominated mathematics and has not lost its value from the time of Newton, and which at the present time flourishes with renewed vitality.

What some call the "elementary solution" of a differential equation (the engineers call it the "influence function" and the physicists "Green's function") is not really a function but actually something more general, a distribution. This very crucial fact, along with new methods, ideas, and even notation introduced by Schwartz in his theory of distributions, has forced an overhauling of the theory of partial differential equations which has improved considerably because of this activity. There are general methods of obtaining the solutions of various boundary problems, and we can, in particular, make use of the powerful method of the Fourier integral and of its generalization; and all this permits us to obtain precise information with reference to the behavior of the solutions. It is possible to say, in general, that a problem set in terms of a system of differential linear equations is solved theoretically. This does not happen, certainly, if the equation is not linear; and in spite of signal progress made in special cases, the truth of the matter is that, in the face of a system of nonlinear partial differential equations, we are helpless, since we do not have a general solution and, what is even worse, we do not in this instance have the almighty arm of the Fourier integral to lean on.

It is a fact that many important cases in physics and technology are governed by nonlinear equations. The venerable mechanics of Newton is not a linear subject. The equation of oscillation and the equation of Van der Pol are not linear. The equations of hydrodynamics with their fearful quadratic terms are not linear.

For these cases, numerical methods offer the only possible solutions, solutions that are most painful to obtain especially where systems of several variables are involved.

It is easy to appreciate the fact that in such instances the electronic computers become an invaluable aid. Computers are used today on a large scale for calculations and automatic control of missiles and rockets, for automatized navigation, for the integration of equations arising from fluid mechanics (in particular, in problems relating to shock waves), for the solution of nonlinear equations arising in the theory of reactors, for the integration of equations in magnetic-hydrodynamics, etc., and of course, a fortiori, for the solution of linear equations, especially those cases in which there are a great number of equations and parameters (systems of equations in theoretical chemistry, linear or quadratic programming). This is a good example of mathematico-technological interaction in every way. There is more, however. The slow progress in nonlinear theories can be explained, in part, by the fatally vicious circle in which they are imprisoned: Since there are no general solutions at hand, the effective calculation for particular cases is extremely difficult and the lack of solutions in particular cases forces us to rely on intuition, unarmed and helpless, without power to create general methods on the basis of inferences and generalizations.

Electronic calculators, in permitting large-scale experimentation with particular cases, have become in fact a very valiant auxiliary of mathematical invention.

These applications of the electronic computers to the solution of mathematical and scientific problems are not, sensu stricto, the only, nor the most important, manifestations of automatization, which is destined in a short time to influence many phases of human activity.

This process of progressive automatization, which is less spectacular than the dramatic events connected with the birth of controlled nuclear energy, is perhaps the most characteristic of the technological age. And again mathematics appears, and in a spectacular role. My thesis is that the deep-rooted reason for the success of automation is to be found in the increasing mathematization of disciplines or bodies and groups of phenomena which even recently were considered too complex to be subjected to rationalization and mathematical method, or even, as many believed, not capable by nature of being mathematized.

A surprising example of this phenomenon is the work by the genius von Neumann on the theory of games. If I were to ask a layman in mathematics if he thought that it was possible to work out a theory for playing poker, or ombre, or bridge "mathematically," i.e., by the "optimum" procedure, it is quite probable that the answer would be an emphatic "No," on the grounds that the infinite complexity of the mental struggle that accompanies any such game would never be reducible to a basic scheme of a group of formulae.

This argument is by no means a new one. It has been used many times, under different guises, to deny to mathematics any relevancy in the field of psychological phenomena; it is often claimed that mind is solely "qualitative," whereas mathematics can be triumphant only over "quantity," etc., etc., etc.

This argument is not capable of "proof" and it is not worth contradiction in the form of another argument in the same category. Von Neumann overcame it by a constructive method: by showing, in his famous paper of 1928, a theory that fully achieves these desiderata. In that period, only a small group of specialists were interested in it. In contrast, the book he wrote sixteen years later in collaboration with Morgenstern evoked immediate reaction. This second book also contained an interesting attempt to provide a mathematical foundation for the science of economics and of politics in the theory of games.

Interest in the book grew even more intense when another important and more frequently used economic discipline, linear programming, was shown to have a connection with game theory. Shortly thereafter the great statistician Abraham Wald developed his famous decision theory, taking as his mathematical model an infinite game. (In Wald's model, nature has the role of one of the players.)

The theory of games, linear and quadratic programming, economic theory, theory of decision, all are mathematical theories. And the list is not complete. It lacks operations research, theory of directions, mathematical linguistics, and many others more novel and disconcerting than these, to which I shall refer shortly.

Confidence in mathematical reasoning and the profound conviction that all problems can be approached advantageously from the point of view of mathematics—this extreme preoccupation with mathematization is one of the specific characteristics of our time.

This is the moment to terminate my list of interactions, but before doing so I must fill a gap in my exposition. It is appropriate to ask this question: If it is true that automatization and automatic control already play an outstanding role in our technological society, how is it possible that in this era of mathematization, these two categories of phenomena have not been mathematized? How is it that there are no theories of control and automatization?

The answer is that theories of control do exist. The cybernetics of the famous Wiener is nothing else, and in it we can glimpse a future theory of robots. In this theory the necessary ingredients will be composed partly of modern logic, the logic of Russell, of Whitehead, of Turing, and of Stone; the remainder will be made up of the very original creation of Claude Shannon, in other words, of information theory.

Cybernetics as well as information theory is based on statistics; ergodic theorems are of great importance for both, and for this reason both are parts of stochastic processes.

Moreover, I ask myself, why not let our imaginations take over and think of the possibility that logic will be governed sooner or later, under the dominion of the calculus of probabilities? Why, I repeat, should we not speculate on the potentialities of a statistical logic?

My concept of the theory of robots takes the following form: It is a statistical theory of artificial intelligence. Its exploitation will lead to one of the greatest creations of this new golden age of mathematics.

So be it!

*MODERN APPLICATIONS OF MATHEMATICS**

Professor Enrique Cansado (Chile)

> "... much of the best mathematical inspiration comes from experience...."
>
> —John von Neumann [1]

> "The most painful thing about mathematics is how far away you are from being able to use it after you have learned it."
>
> —James R. Newman [2]

Although it is difficult to reach agreement as to the significance of the adjective "modern," I shall use the word to describe movements that belong principally to the last quarter of the century. During this twenty-five-year period, many interesting and widely differing models and theories based on mathematics have been vigorously developed; an intense and diversified interest in mathematical applications has also been created; and the utilization of mathematical methods has invaded new scientific and technological fields (I refer principally to economics, social sciences, business administration, etc.).

Historically, a great many of these new interests in mathematical applications have been associated with the birth, during the clamor of World War II, of the so-called "operations research." If, in the beginning, this application of mathematics was merely a scientific investigation of military operations, it quickly grew into a scientific investigation of any operational system, as, for example, a manufacturing plant, the economy of a nation.

In "operations research of a system," it is necessary to assume, among other things, the formulation of a "mathematical model," as, for example, a system of equations, inequations, or differential equations, which expresses the relations among the variables present in that system. Such a "model" generally contains a certain number of constants, the values of which are not specified,

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*Translated from the Spanish; some sections omitted. For the bibliography of this address, see the Appendix.

[1] John von Neumann, "The Mathematician," The World of Mathematics, Vol. IV, ed. by James R. Newman, New York: Simon and Shuster, 1956, p. 2059.

[2] James R. Newman, "Commentary," op. cit., Vol. III, p. 1978.

or a number of parameters that are to be statistically estimated on the basis of data on hand or otherwise obtainable from direct observation of the system under analysis.

The utilization of mathematical models of this type generally entails laborious procedures of the numerical calculus which have created a need for automatic electronic machines. Without these machines (also modern) it would have been impossible to achieve all the great advances in modern mathematical applications.

In the background of all these developments, it should be noted that, as a decisive factor in the applicability of mathematics to extremely complex situations similar to those which previously confronted the natural sciences, there lies a mathematical theory which has not been in existence long—the "Calculus of Probability and Mathematical Statistics." Their models, not as rigid as those of the classical mathematics, or deterministic, fit very easily and realistically in situations in which risk and uncertainty are unavoidable and important aspects of the problem.

What follows is intended to be a brief synthesis of descriptions of the principal methods, theories, and problems that will cover the topic considered to be the main one of this address. Since the theories and applications are at the present time in the process of being developed, it will be impossible to "hit the mark" rigorously and systematically in this presentation. I do not intend to give the impression that my exposition will be objective and accurate; preferences and personal judgment will be in evidence throughout in the selection of some of the topics and sequences, and in the treatment given to each of these. To use an analogy, it might be said that we are too near the trees to see the forest.

LINEAR PROGRAMMING

A progenitor and a birth date are usually assigned to linear programming. They are George B. Dantzig (U.S.A.) and 1947. An unforeseen aspect of the "cold war," however, has provided paternal priority in the person of the eminent mathematician L. V. Kantorovitch and has moved back the date of the birth of the theory to 1939 (see [49] in the bibliography in Appendix). In a recent academic lecture, the mathematician J. Rey Pastor [65] informed us that the very first problem in linear programming was formulated by Monge in 1776, and that Fourier had already studied systems of linear inequalities in 1823. Recognition as work following that of Fourier is also given to the work of F. L. Hitchcock [47] on a transportation problem (1941) and to that of G. J. Stigler [70] on a diet problem of a minimum cost (1945). That mathematicians were interested in the theory of linear inequalities and convex polyhedra (which is closely related to the theory of linear programming) is indicated by the works of the following: J. Farkas [35], 1902; H. Weyl [82], 1935; and T. S. Motzkin [62], 1933.

These historical explanations do not modify the fact that the burgeoning use of this method can be traced to the Dantzig article, published in 1951, in which a general numerical method of solution by the simplex method is presented for a problem in linear programming. Interest in the method and applications of it, both numerous and different, were immediately in evidence. Even today, it can be said (in spite of the participation in and collaboration of many eminent writers in this work) that G. B. Dantzig is the main creator of the methods and new developments in linear programming.

The following is a typical problem in linear programming.

Find the maximum of the linear function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n \tag{1.1}$$

where $c_1, c_2, c_3, \ldots, c_n$ are known constants and where $x_1, x_2, \ldots, x_n$ are variables which are subject to the following restrictions (linear inequations):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \end{cases} \tag{1.2}$$

where the constants $a_{11}, a_{12}, \ldots, a_{1n}; a_{21}, a_{22}, \ldots, a_{2n}; \ldots; a_{m1}, a_{m2}, \ldots, a_{mn}; b_1, b_2, \ldots, b_m$ and the variables, the x's, are subject to a condition of being non-negative, i.e.,

$$x_1 \geq 0, \; x_2 \geq 0, \; \ldots, \; x_n \geq 0 \tag{1.3}$$

Using the matrix (of m rows and n columns),

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the row vector (of n components)

$$\vec{c} = [c_1, c_2, \cdots, c_n],$$

and the two column vectors (of m and n components, respectively),

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

the same problem can be formulated as follows:

Find the non-negative vector $\vec{x}$ that makes a maximum of the expression

(1.1') $\vec{c}\ \vec{x}$

and satisfies the inequality

(1.2') $A\vec{x} \leq b$,

where c, A, b are obtained from given data.

An analogous problem can be set up for minimizing a linear function.

In other problems of linear programming, one, several, or all of the restrictions may be equations instead of inequations. It should be noted that an inequation is equivalent to two inequalities, since $R \neq S$ is equivalent to $R > S$ or $R < S$.

Finally, there are problems in linear programming in which one, several, or all of the variables may be negative. It should be noted that any variable x which is not limited to the condition $x \geq 0$ can always be expressed as the difference of two non-negative variables, i.e., $x = x' - x''$, where $x' \geq 0$, $x'' \geq 0$.

It should be observed, however, that finding the minimum of

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

is equivalent to finding the maximum of

$$-z = -c_1x_1 - c_2x_2 - \dots - c_nx_n.$$

Therefore any problem in linear programming can be expressed as a problem of maximation.

The simplex method was originally proposed by G. B. Dantzig (pursuing an idea, as he himself has explained, that was suggested by von Neumann during a conversation). It consists essentially of a modification and adaptation of Jordan's method. (This is essentially an adaptation of the well-known method of Gauss [69], [17], for a solution of a system of n linear equations with n unknowns. Such a simplex method gives us a first "feasible solution," i.e., a set of values of $x_1, x_2, \dots, x_n$ which satisfy the conditions (1.2) and (1.3), or analogous conditions for the minimizing problem. Generally this first solution does not give the maximum (minimum) value of the objective function. If this is the case, we look for another, generally a better one, i.e., to another "program" or solution, generally an improved solution that gives a closer approximation than the first. From the second solution, we proceed to a third, and then to a fourth, and so on, until we get a solution that cannot be improved. The simplex method indicates when this has occurred and the result is established as the "optimal solution" of the problem.

As we can see, the simplex method is an "iterative method" which, in a finite number (not known beforehand) of steps or "interactions," permits us to find the solution of the problem. Each one of these steps involves only elementary operations: addition, subtraction, multiplication, and division (of positive and negative numbers), and these steps or interactions are also subjected to some elementary assumptions upon which the strategy of the new solution will be based. All this work can be entrusted to an automatic electronic calculator by means of a routine or a program, the formulation of which presents no difficulty. If a desk calculator is employed (e.g., Friden, Marchant), the operator can easily be taught the routine of the simplex method described above.

The set of feasible solutions is a "convex polyhedral set," a property discovered by Fourier in 1823, in which the "basic" feasible solutions (always a finite number) correspond to the vertices or extreme points of the set. The convexity of this set is a guarantee that the maximum (minimum) of the linear function is at one of these vertices. The Dantzig simplex method consists in passing from one vertex to an adjacent one, moving along an edge, until the maximum (minimum) of the objective function is reached.

It is possible for the set of feasible solutions to be an empty set. This would mean that the system of inequalities (1.2) and (1.3) is an incompatible or inconsistent system. In this case, the problem has no solution. On the other hand, there are many problems in linear programming for which the set of feasible solutions is not a bounded set. When passing through feasible solutions, the objective function increases (decreases) indefinitely, and strictly speaking, the function would not assume a maximum (minimum) value.

There are problems of linear programming in which the optimum (or optimum program) is unique. There are others with an infinite number of optimum solutions. By means of the simplex method, the "basic" optimum solutions (optimum vertices), always finite in number, can be found. From these, the infinite number of optimum solutions can be obtained. Suppose, for example, that the basic optimum solutions are

$$x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}; \quad x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}; \quad \ldots;$$

$$x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}$$

Then the following are optimum solutions:

$$\begin{cases} x_1 = t_1 x_1^{(1)} + t_2 x_1^{(2)} + \cdots + t_r x_1^{(r)} \\ x_2 = t_1 x_2^{(1)} + t_2 x_2^{(2)} + \cdots + t_r x_2^{(r)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = t_1 x_n^{(1)} + t_2 x_n^{(2)} + \cdots + t_r x_n^{(r)} \end{cases}$$

where

$$t_1 \geq 0,\ t_2 \geq 0,\ \ldots,\ t_r \geq 0,\ t_1 + t_2 + \ldots + t_r = 1$$

In other words, the linear combinations of the r basic solutions are optimum solutions. In graphic (or geometrical) terms, it can be said that if there are two optimum vertices ($r = 2$), then all points on the line joining the two vertices are also solutions, and, in general, if there are r optimum vertices, the infinite number of points on the "faces" of the polyhedral set determined by the r vertices are also optimum solutions. (Such a set of optimum solutions constitutes the "basic convex polyhedron" which is generated by the r optimum solutions.)

The possibility of obtaining the basic optimum solutions can be realized by means of the simplex method combined with topological results of the "labyrinth" problems.

There is a chance that in the application of the simplex method, a theoretical difficulty may arise. (It has not as yet occurred in a practical problem.) This problem is related to the so-called "degenerate case," or the "case of cycles," a case which happens when, in applying the simplex method, there is no improvement in the solution, i.e., the procedure repeats a basis, thereby completing a cycle. In such a case, moreover, the automatic computer will repeat the cycle indefinitely without obtaining an optimum solution. This difficulty was resolved by Charnes [18], and, independently, with a slight modification (perturbations technique) of the simplex method, by Dantzig, Orden, and Wolfe [30].

Problems in linear programming can also be studied in pairs of dual problems, with each couple composed of a problem in maximization and one in minimization.

In "classical" mathematics there is a problem of finding the maximum (minimum) of a function of n variables which satisfy m restrictions in the form of equations. The solution involves the use of the so-called "Lagrange multipliers" as well as the "Lagrange function," or "Lagrangian." Partial derivatives taken with respect to each of the n variables of the problem along with the m restrictions give a system of $(n + m)$ equations, the solution of which yields the values of the required variables, and, incidentally, the values of the m Lagrange multipliers (which had not been used to any great extent before the introduction of this type of problem). In this classical problem, there is no restriction of linearity on the function. In this connection, linear programming may be regarded as a special case of the classical one. On the other hand, however, linear programming involves inequalities as well as equalities and, in this sense, it represents a fundamental extension or generalization of the classical problem. It may be observed also that all equalities can be expressed in terms of inequalities, but that it is not possible to express an inequality in terms of an equality.

Nevertheless, the method of Lagrange multipliers can be applied (with some modification) to the analysis of problems in linear programming, as well as to nonlinear programming. In effect, given the following problem:

To find the maximum of z, where

$$z = \vec{c}\,\vec{x}$$

for

$$\vec{b} - A\vec{x} \geq 0$$
$$\vec{x} \geq 0$$

Consider the negative row vector

$$y = [y_1, y_2, \dots, y_m]$$

whose m components will be called Lagrange multipliers. The Lagrange function, a bilinear form, will be defined as

$$\Phi(\vec{x}, \vec{y}) = c\vec{x} + y\,(\vec{b} - A\vec{x})$$

We will say that $\vec{x}_0$ and $\vec{y}_0$ determine a "saddle point" of $\Phi(\vec{x}, \vec{y})$ if

$$\vec{x}_0 \geq 0,\ \vec{y}_0 \geq 0, \text{ and } \Phi(\vec{x}, \vec{y}_0) \leq \Phi(\vec{x}_0, \vec{y}_0) \leq \Phi(\vec{x}_0, \vec{y})$$

for all $\vec{x} \geq 0$ and $\vec{y} \geq 0$.

(It should be observed that at point $(\vec{x}_0, \vec{y}_0)$, the function $\Phi(\vec{x}, \vec{y})$ has a maximum with respect to $\vec{x}$, and a minimum with respect to $\vec{y}$. For this reason, it is sometimes designated as a "maximin" or a "minimax," and

$$\Phi(\vec{x}_0, \vec{y}_0) = \max_x \min_y \Phi(\vec{x}, \vec{y}) = \min_y \max_x \Phi(\vec{x}, \vec{y})$$

The fundamental theorem, which follows, can be demonstrated by use of the theory of convex cones (polyhedrics).

Fundamental theorem: The vector $\vec{x}_0$ is an optimum solution of the above problem if and only if there is another vector $\vec{y}_0$ such that $(\vec{x}_0, \vec{y}_0)$ is a saddle point of the Lagrange function $\Phi(\vec{x}, \vec{y})$. A dual of this problem is:

To find the minimum of w, where

$$w = \vec{y}\vec{b}$$

for

$$\vec{y}A - \vec{c} \geq 0$$
$$\vec{y} \geq 0$$

with $\vec{x}$ a non-negative column vector having components $x_1, x_2, \ldots, x_n$ (Lagrange multipliers) and Lagrange function

$$\psi(\vec{y}, \vec{x}) = -\vec{y}\vec{b} + (\vec{y}A - \vec{c})\vec{x}$$

It can be proved that

$$\psi(\vec{y}, \vec{x}) = -\Phi(\vec{x}, \vec{y})$$

and, therefore, the saddle point problem is equivalent to an analogous one in linear programming.

It should be noted that although theoretically the data used in a programming problem are known constants, in actual practice this normally is not the case. To solve a problem, one usually has to begin by estimating the values of such constants. This is a difficult matter and the validity of such estimates is always limited. There is an interesting way of studying the effects of this unavoidable incertitude on the optimum program. There are two points of view that dominate this type of analysis:

1. The "deterministic" point of view. (See references [36], [41], and [75].)
2. The "probabilistic" point of view which employs stochastic processes. (See [75].)

NONLINEAR PROGRAMMING

A natural extension of linear programming is to be found in the maximation (or minimation) of a nonlinear function of n variables when those variables satisfy m nonlinear inequalities:

Example: To find the maximum of $z = f(x_1, x_2, \ldots, x_n)$ bounded by

$$(2.1) \quad \begin{cases} g_1\ (x_1, x_2, \cdots, x_n) \geq 0 \\ g_2\ (x_1, x_2, \cdots, x_n) \geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_m(x_1, x_2, \cdots, x_n) \geq 0 \\ x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0 \end{cases}$$

The classical problem of bounded maxima or conditioned maxima is:

Find the maximum of $z = f(x_1, x_2, \dots, x_n)$ for

$$(2.2)\quad \begin{cases} g_1(x_1, x_2, \cdots, x_n) = 0 \\ g_2(x_1, x_2, \cdots, x_n) = 0 \\ \dots\dots\dots\dots\dots\dots \\ g_m(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

To solve this, we use the Lagrange function:

$$(2.2')\quad \Phi(\vec{x}, \vec{y}) = f(\vec{x}) + \sum_{i=1}^{m} y_i g_i(\vec{x})$$

where $y_1, y_2, \dots y_m$ are the so-called Lagrange multipliers.

The solution of the problem is obtained by solving the following system of $(m+n)$ equations:

$$\frac{\partial \Phi(\vec{x}, \vec{y})}{\partial x_j} = 0, \quad (j = 1, 2, \dots, n)$$

$$\frac{\partial \Phi(\vec{x}, \vec{y})}{\partial y_i} = 0, \quad (i = 1, 2, \dots, m)$$

with $(n+m)$ unknowns: the n variables x_j and the m multipliers y_i. (Note that if

$$\frac{\partial \Phi(\vec{x}, \vec{y})}{\partial y_i} = 0,$$

it implies that $g_i(x_1, x_2, \dots, x_n) = 0$. Notice that the y_i's can be positive, negative, or null.)

H. W. Kuhn and A. W. Tucker [53] modified Lagrange's classical method in order to apply it to a programming problem where $f(x_1, x_2, \dots, x_n)$ and all the functions $g_1(x_1, x_2, \dots, x_n)$ are "concave" functions (with an additional condition).

It is said that $f(x_1, x_2, \dots, x_n)$ is a "concave" function if

$$f(ax_1' + bx_1'', ax_2' + bx_2'', \dots, ax_n' + bx_n'') \geq$$

$$af(x_1', x_2', \dots, x_n') + bf(x_1'', x_2'', \dots, x_n'')$$

for all $a > 0$, $b > 0$, $a + b = 1$ and for any $x_1', x_2', \dots, x_n'$, and $x_1'', x_2'', \dots, x_n''$. If the sign $\geq$ is replaced by $\leq$, we have the definition of the "convex" function. If the sign $\geq$ is replaced by $>$, we have the definition of the "strictly concave" function, and so on. It is clear that if $f(x)$ is concave, $-f(x)$ is convex. Notice that any linear function is concave and also convex, but it is not "strictly concave" nor "strictly convex." (Note that $\Phi(\vec{x}, \vec{y})$ is, in this case, concave with respect to $\vec{x}$ and convex with respect to $\vec{y}$, although not "strictly" in the last case.)

With the above conditions, Kuhn and Tucker proved that problem (2.1), of the concave programming problem, is equivalent to the problem of finding a saddle point of the Lagrange function. We say that $\vec{x}_0$ is the optimum solution of problem (2.1) if and only if there is a $\vec{y}_0$ such that

$$\Phi(\vec{x}, \vec{y}) \leq \Phi(\vec{x}_0, \vec{y}_0) \leq (\vec{x}_0, \vec{y}) \text{ for all } \vec{x} \geq 0, \vec{y} \geq 0$$

and for which $\vec{x}_0 \geq 0, \vec{y} \geq 0$.

In the case where the functions f and g_i have partial derivatives, Kuhn and Tucker proved that the programming problem, or the equivalent problem of finding a saddle point of $\Phi(\vec{x}, \vec{y})$, is equivalent in fact to the problem of solving the following system of inequations:

$$(2.1') \quad \begin{cases} \dfrac{\partial \Phi(x,y)}{\partial x_j} \leq 0, \quad j = 1, 2, \ldots, n; \quad \dfrac{\partial \Phi(x,y)}{\partial y_i} \geq 0, \quad i = 1, 2, \ldots, m \\ \sum\limits_{j=1}^{n} x_j \dfrac{\partial \Phi}{\partial x_j} = 0; \quad \sum\limits_{i=1}^{m} y_i \dfrac{\partial \Phi}{\partial y_i} = 0 \\ x_j \geq 0, \; j = 1, 2, \ldots, n; \quad y_i \geq 0, \; i = 1, 2, \ldots, m \end{cases}$$

It can therefore be said that the classical problem of bounded or conditioned maxima (2.2) can be reduced to the solution of a system of equations (2.2'); and the modern problem of concave programming (2.1) reduces to the solution of a system of inequations (2.1').

The development of algorithms and numerical solutions for problems in nonlinear programming is in full swing today and we can hope for important results in the future.

A general algorithm, applicable to any concave problem and proposed by K. J. Arrow and L. Hurwicz [1], is the so-called "gradient method" that permits us to obtain an approximate solution by means of a system of differential equations. Another is the so-called "Newton's method" proposed by Cheney and Goldstein [21] for a convex program (i.e., a minimation of a convex function).

When the functions g_i are linear and the function f is a quadratic, we say that the problem involves "quadratic programming." There are several algorithms (based on extensions and modifications of the simplex linear program method) for solutions of this type, some of which were proposed by the following: Wolfe [84], Markowitz [57], Houthakker [48], Barankin and Dorfman [4], Frank and Wolfe [38], Hildreth [46], Hartley [44].

If f is concave and, moreover, if $f(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$, the function f is said to be "separably" concave. Charnes and Lemke [20] proposed a numerical procedure for solving this type.

DYNAMIC PROGRAMMING

This involves a technique attributable principally to Richard Bellman of the RAND Corporation (Santa Monica, California), [9]. Most of the works published on this subject are the works of Bellman and his colleagues at the RAND Corporation.(See [7, 8, 10, 11, 12].)

GAME THEORY

The mathematical theory of "strategy games" refers to situations in which there is a contest between two or more persons whose interests are, to some extent, in conflict. The result of such a situation, conflict, or game depends only partially on the players who engage in the contest. It depends also, in some small part, on luck, but principally on the ability and intelligence of the participants. The latter feature is the one that distinguishes this type of game from the so-called "hazard" games, like the lottery, roulette, etc., where the "luck" of an idiot can exceed that of a genius. On the other hand, in chess (evidently a game of strategy), a beginner has no chance against an international champion.

The theory of games aroused a deep and far-reaching interest when von Neumann and Morgenstern [80] first presented it in 1944 as a method of analyzing conflicts of interest that are always present in the sphere of economics. Applications of the theory to fields other than economics, e.g., social problems, both political and military, were immediately forthcoming, and they became a very important tool for purposes of operations research.

Before the appearance of the von Neumann and Morgenstern book, mathematicians had already been considering strategy games. Evidence of this interest is to be found in the publications of Zermelo [86], 1912, and Borel [14], 1921, both of whose writings include a demonstration (for two particular cases) of the fundamental theorem of games (now called the "minimax" theorem), and also a conjecture that the theorem was generally false; and in the truly fundamental work of von Neumann [78], 1928, in which he proves the minimax theorem, and of von Neumann [79] in 1937, and of de Ville [77], in 1938.[3]

In game theory, it is assumed that each player makes a selection from among several alternatives (not necessarily the same alternatives for each player or the same number of alternatives); that simultaneously each player makes a selection from among the alternatives, with no player knowing what the other players have selected; and finally, once the selections of the players are known (at the end of each play), each player obtains a "result" or "profit" that can be positive (effective gain), negative (loss), or zero (draw). Such results or profits depend upon the choices of each one of the players.

The concept of "strategy" is a fundamental one in the theory of games. By strategy is meant the rule that the player applies in order to choose one of the alternatives in each play.

Let us consider the simplest case, the "zero-sum, two-person" game. This game is limited to two persons, two teams, two companies, or to two nations, etc. The interests of the participants are diametrically opposite, because what one wins, the other loses. (The sum of the "profits" of the players is zero; i.e., positive profit of one player + negative profit of the other player = 0, hence the term "zero-sum" in describing the game.) Moreover, suppose that player J_1 has at his disposal m alternatives $A_1, A_2, A_3, \ldots, A_m$, and that player J_2 has n alternatives $B_1, B_2, B_3, \ldots, B_n$. The outcome of each game depends upon the alternative A_i chosen by J_1 and the alternative B_j chosen by J_2. These outcomes can be represented by means of the profits for player J_1.

	B_1	B_2	$\ldots$	B_j	$\ldots$	B_n
A_1	a_{11}	a_{12}	$\ldots$	a_{1j}	$\ldots$	a_{1n}
A_2	a_{21}	a_{22}	$\ldots$	a_{2j}	$\ldots$	a_{2n}
	....	....	....	....	....	....
A_i	a_{i1}	a_{i2}	$\ldots$	a_{ij}	$\ldots$	a_{in}
	....	....	....	....	....	....
A_m	a_{m1}	a_{m2}	$\ldots$	a_{mj}	$\ldots$	a_{mn}

The matrix $A = [a_{ij}]$ of m rows and n columns is called a

[3] See Econometrica, Vol. 21, 1953, pp. 95-127, for articles in English translation, by Borel, as well as by Frechet and von Neumann, on historical questions and priority in the theory of games of strategy.

"profits matrix" (for player J_1). The profits for J_2 are evidently $-a_{ij}$. Player J_1 is said to be the "maximizing" player (because he wishes to "receive" a maximum a_{ij}) and player J_2 is the "minimizing" player (since he wishes to "pay" a minimum a_{ij}).

The strategy, or rule for making decisions, employed by J_1 cannot be a deterministic one since J_2 would seek advantage by finding out what it is. (This can be done if J_2 spies on J_1 or observes what J_1 does in a succession of moves.) The strategy used by J_1 will be probabilistic in nature, i.e., J_1 selects one of the alternatives open to him in each play by "casting lots." The strategy of J_1 can therefore be specified by the probabilities of the m alternatives resulting from this casting of lots, i.e., $x_1, x_2, x_3, \ldots, x_m$. One strategy of J_1 is a specific set of values of these probabilities. These values must satisfy the following conditions:

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0; \quad x_1 + x_2 + \ldots + x_m = 1$$

and they are generally arbitrary. Note that J_1 has an infinite number of strategies at his disposal.

Analogous considerations apply to J_2, who also has an infinite number of strategies available, each having the form $y_1, y_2, \ldots, y_n$, where

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_n \geq 0; \quad y_1 + y_2 + \ldots + y_n = 1$$

It is evident that when J_1 employs strategy $\vec{x} = \{x_1, x_2, \ldots, x_m\}$ and J_2 the strategy $\vec{y} = \{y_1, y_2, \ldots, y_n\}$, the profits of J_1 will fluctuate from game to game and will depend upon the lots cast by the two players before each move.

An analysis can not be provided for each move of the game but only for the game considered as a set of all possible moves. This can be done by considering the "average gain for each move" that J_1 receives. This profit can be represented as follows:

$$G(\vec{x}, \vec{y}) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j$$

The function $G(\vec{x}, \vec{y})$ is called the profit function of J_1.

It is evident that J_1 will select his strategy $\vec{x}$ in such a way that $G(\vec{x}, \vec{y})$ will be large (maximizing player), and that J_2 will select a strategy that will make $G(\vec{x}, \vec{y})$ small (minimizing player). Both players pursue their objectives through strategies $\vec{x}_0$ and $\vec{y}_0$ that satisfy the following conditions:

$$G(\vec{x}_0, \vec{y}) \geq G(\vec{x}_0, \vec{y}_0) \geq G(\vec{x}, \vec{y}_0)$$

for all $\vec{x}$ and all $\vec{y}$.

Actually J_2 doesn't have a better choice, because if he should use some other strategy $\vec{y}$, J_1 would get a larger profit by using his own optimum strategy $\vec{x}_0$. By the same argument, J_1 doesn't have a better choice than $\vec{x}_0$, because if he were to select another strategy $\vec{x}$, J_2 would get a smaller profit by using his own optimum strategy $\vec{y}_0$.

It should be observed that $\vec{x}_0, \vec{y}_0$ constitute a saddle point of the profit function. The following can be proved:

To guarantee the existence of

$$\max_{\vec{x}} \min_{\vec{y}} G(\vec{x}, \vec{y}) \quad \text{and} \quad \min_{\vec{y}} \max_{\vec{x}} G(\vec{x}, \vec{y})$$

satisfying the condition that

$$\max_{\vec{x}} \min_{\vec{y}} G(\vec{x}, \vec{y}) = \min_{\vec{y}} \max_{\vec{x}} G(\vec{x}, \vec{y})$$

it is necessary and sufficient that $G(\vec{x}, \vec{y})$ have a saddle point $\vec{x}_0, \vec{y}_0$. Then it would also be true that

$$G(\vec{x}_0, \vec{y}_0) = \max_{\vec{x}} \min_{\vec{y}} G(\vec{x}, \vec{y}) = \min_{\vec{y}} \max_{\vec{x}} G(\vec{x}, \vec{y})$$

Von Neumann demonstrated in 1928 (using the fixed point topological theorem of Brouwer) that "all profit functions $G(\vec{x}, \vec{y})$ satisfy the above theorem and therefore have saddle points." This famous minimax theorem guarantees the existence of an optimum strategy for both players. The point $(\vec{x}_0, \vec{y}_0)$ constitutes a solution of the "game," and $v = G(\vec{x}_0, \vec{y}_0)$ is the value of the "game."

Later, Ville [77], Karlin [50], and others gave new proofs of the minimax theorem based on the theory of convex polyhedra. (See [13], [42], [55], and [82].)

It is easy to show that if the same quantity k is added to each of the elements of the profits matrix $A' = \{a_{ij} + k\}$, we obtain another game with the same solution as the original and a value $v' = v + k$. By choosing k large enough, it is possible to get non-negative elements in the matrix so that v' is guaranteed to be positive.

If J_2 uses strategy $\{0, 0, \ldots, 1, \ldots, 0\} = \vec{e}_i$, where $\vec{e}$ is a unit vector of n components for which the ith component is 1 and all others are zeros, and if J_1 uses strategy $\{y_1, y_2, \ldots, y_n\} = \vec{y}$, the average profit of J_1 will be

$$G(\vec{e}_i, \vec{y}) = a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{in}y_n$$

and it can be proved that if

$$G(e_i, y) \leq v \quad \text{for } i = 1, 2, \ldots, m$$

then the strategy is an optimum one.

Let us consider the game whose profits matrix is A' and whose value v' (supposed to be unknown) can be taken as positive. Introduce the quantity $u \geq v' \geq 0$. In accordance with the preceding discussion, it happens that we can find the solution to the game by finding a solution to the following problem:

Find the minimum of u, so that

$$\begin{cases} a'_{11}y_1 + a'_{12}y_2 + \cdots + a'_{1n}y_n \leq u \\ a'_{21}y_1 + a'_{22}y_2 + \cdots + a'_{2n}y_n \leq u \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a'_{m1}y_1 + a'_{22}y_2 + \cdots + a'_{mn}y_n \leq u \end{cases}$$

with $y_1 + y_2 + \cdots + y_n = 1$

and $y_1 \geq 0,\ y_2 \geq 0,\ \ldots\ y_n \geq 0$

This problem in turn is equivalent to the following:

Find the minimum of u, so that

$$\begin{cases} a'_{11}w_1 + a'_{12}w_2 + \cdots + a'_{1n}w_n \leq 1 \\ a'_{21}w_1 + a'_{22}w_2 + \cdots + a'_{2n}w_n \leq 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a'_{m1}w_1 + a'_{m2}w_2 + \cdots + a'_{mn}w_n \leq 1 \end{cases}$$

with $w_1 + w_2 + w_3 + \cdots + w_n = \frac{1}{u}$

and $w_1 \geq 0\ ;\ w_2 \geq 0\ ;\ w_3 \geq 0\ ;\ \ldots\ ;\ w_n \geq 0$

and where

$$y_i = w_i \quad \text{for } i = 1, 2, \ldots, n$$

And this, furthermore, is equivalent to finding the maximum of

$$z = w_1 + w_2 + \cdots + w_n$$

with

$$\begin{cases} a'_{11}w_1 + a'_{12}w_2 + \cdots + a'_{1n}w_n \leq 1 \\ a'_{21}w_1 + a'_{22}w_2 + \cdots + a'_{2n}w_n \leq 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a'_{m1}w_1 + a'_{m2}w_2 + \cdots + a'_{mn}w_n \leq 1 \end{cases}$$

$$w_1 \geq 0 \; ; \quad w_2 \geq 0 \; ; \quad \ldots \; ; \quad w_n \geq 0$$

where

$$z = \frac{1}{u}$$

The solution to this problem in linear programming gives the value of the game v with

$$v = v' - k = \frac{1}{z^{(0)}} - k \, ,$$

where $z^{(0)}$ is the maximum value of z and the optimum program is $w_1^{(0)}, w_2^{(0)}, \ldots, w_n^{(0)}$, and the optimum strategy for J_2 is

$$y_i^{(0)} = \frac{w_i^{(0)}}{z_i^{(0)}} \; ; \quad i = 1, 2, \ldots, n$$

The optimum strategy of the other player J_1 can be obtained by an analogous method involving the solution of the dual problem of minimization and bears a resemblance to the kind of problem to which the simplex method is applied.

The equivalence of the two-person game and programming was discovered by Dantzig and von Neumann and has been of benefit to both theories.

The theory of games for number of players $n > 2$ has not been developed satisfactorily. The problem is very much complicated by the possibility of agreements among the players. Games having a profits sum which is not zero also present great difficulties.

All these extensions of the theory of games are being studied, however, by many leading mathematicians.

OTHER APPLICATIONS OF MATHEMATICS

We cannot close this presentation without mentioning three other applications:

Theory of Inventories

This theory produces, through deterministic and probabilistic mathematical models, the optimum strategy of inventory. Inventory is the amount of stock of economic goods that must be maintained to satisfy a variable and uncertain demand. On the one hand, there is a desire to hold down expenses of warehousing. On the other hand, losses in clientele are sustained when demand is not satisfied, which suggests keeping a large inventory. The optimum solution will be, for the time being, a compromise between these contrary forces. (For more detail, see [3], [23], [33], [60], [61], [67], [68], and [76].)

Theory of Queues

This theory concerns the most convenient method of giving adequate service when demand is variable and uncertain. There are queues in front of theaters, queues of airplanes waiting to land, queues of machines to be repaired, letters to be answered, and so on. Queues involve expense and annoyance. To reduce the waiting period expenses must be increased. As in the case of inventories, the solution requires a compromise. Those who are interested in these problems and in the corresponding mathematical theory should consult [23], [33], [60], [61], [66], and [67].

Decision Theory

Finally, we should mention decision theory, even if only by name, as another application of game theory. (See [22] and [71].)

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Discussion Following Address by Professor Cansado

The discussion centered about the following questions: Which of the new applications described by Professor Cansado can be presented at the secondary school level, and in what depth? Should only specialists study these applications, i.e., should they be included in the program of the university? Are there any textbooks on these subjects suitable for use in the secondary schools?

In answer to the first question, Professor Cansado pointed out that the following topics, for example, lend themselves to secondary school teaching: solution of inequalities; systems of inequalities; graphic solution of problems in linear programming; concept of half plane; convex polyhedra. This response elicited from two speakers the suggestion that there were other topics of greater priority for the secondary school student, and, in connection with the related question about inclusion of these topics in the university program, one speaker expressed the opinion that linear programming was too elementary for that level and should be learned "on the job" by the specialist.

Professor Cansado defended his thesis against the implication that the theory of inequalities, saddle point, and convex functions should be classified as "for the specialist," stating that numerical methods, the great tools of the physicist, are being avoided by engineering students who have great need of them. Linear programming and allied topics would serve as an introduction to the powerful numerical procedures for this type of student.

The question concerning secondary school textbooks on these topics brought from Professor Bundgaard a response to the effect that there is a book on polygonal inequations for the Danish secondary schools. Professor Cansado contended that mathematics teaching is one hundred years behind research, and that we do not have one hundred years to "catch up"; therefore we should be prepared to write the texts needed.

Except for expressions of agreement with Professor Cansado, there was no further discussion.

REFORM OF INSTRUCTION IN GEOMETRY

Professor Howard F. Fehr (U.S.A.)

The study of geometry at both the secondary and the university level presents some of the most controversial issues in mathematics instruction today. The widespread dissatisfaction with the curriculum stems in part from the advances in mathematics that have influenced our understanding of what constitutes the discipline that we classify as "geometry." Before considering a program of reform, therefore, it is necessary to examine the movements that have shaped our present view of the subject and gauge their effect on our teaching.

DEVELOPMENT OF GEOMETRY

The role played by the discovery of the non-euclidean geometries in freeing the subject from the yoke of intuition is too well known to deserve more than passing attention. Not so well known, perhaps, is the fact that as far back as the sixteenth century, Clavius pointed to a gap in Euclid's treatment of proportion.[1] A century later Leibnitz drew attention to Euclid's neglect to provide for the intersection of two circles, each of which passes through the center of the other. The dependence of several proofs in Euclid upon the unstated assumption of the order of points on a line (betweenness) was noted by Gauss. This last-named omission acquired added significance when it became the focal point of the movement to "save" Euclid by ensuring completeness of the set of axioms.

Notwithstanding the criticism that had been directed at Euclid, the only geometry up to 1800 continued to be Euclid's geometry. This situation, however, changed greatly in the nineteenth century. There was a burst of activity in the subject, with emphasis at first being placed on the development of different types of geometry. This movement reached a climax in the publication by Riemann of "The Hypotheses Which Lie at the Foundations of Geometry."[2] A new light was thrown on all these different types of geometry by Felix Klein,[3] who used the theory of groups of transformations applied to

[1] This was the failure by Euclid to prove the existence of a fourth proportional to three given magnitudes.

[2] Written and delivered in 1854, but published in 1868. Dealt with n-ply or n-dimensional spaces with metrics.

[3] "Erlangen programm," announced in 1872.

manifolds in classifying the various geometries. Although the so-called "Erlangen programm" of Klein had the effect of showing the essential unity of all branches of geometry developed up to that time, we know today that it has severe limitations, especially when considered in connection with metric, vector, and topological spaces.

The development of geometry in the last quarter of the nineteenth century is characterized by two distinct trends. One of these (differential geometry) managed to draw geometry so completely into the domain of analysis that the subject is now studied from that point of view, and the word "geometry" used in this connection takes on the aspects of an adjective rather than a noun. The other trend had as its motivating force the restoration of the prestige which had been accorded Euclid's geometry down through the ages. The aim of the group of mathematicians who gave substance to this trend was to again make it possible to regard geometry as a set of propositions proceeding in the orderly fashion that characterized the Euclid, that is, by the laws of logic, from a set of axioms.

To Pasch goes the credit of having been the first to give an abstract formulation of a complete set of axioms. This initial work was followed in short order by that of Peano, Pieri, and other members of the "Formulaire" group. (It should be noted that the writers responsible for the "Formulario" formed an 1890 counterpart of our present-day Bourbaki, in that they published avant-garde mathematics under one name.)

The period of the perfection of Euclid culminated in 1899 with the famous Grundlagen der Geometrie of Hilbert, which not only is better known than the other formulations of geometry, but has had a greater influence on the teaching of geometry at all levels. The work of Hilbert, appearing so appropriately at the threshold of the new century, cut the last strings that tied geometry to intuition, and gave testimony to the fact that, so far as mathematicians were concerned, formalism had won a signal victory.

TEACHING OF GEOMETRY

In the first decades of this century, the movement to refine the axiomatic basis of Euclid had little or no effect on the teaching of geometry at either the secondary or the collegiate level. The only change in the secondary school geometry of this period that may be ascribed to this movement is the change in the textbooks of the definition of axiom from a "self-evident truth" to "a statement to be accepted without proof." Beyond this, Euclid, with all its omissions, prevailed. At the collegiate level, courses in geometry bore the title, "Synthetic Projective Geometry," "Analytic Geometry," or

"College Geometry." The subject matter of the last-named course covered such topics as Brocard points, Simson line, Ceva's theorem, etc., all presented in the classical style of Euclid.

This situation of inactivity changed in the 1930's with the revival of interest in the Hilbert axioms as a suitable basis for an instructional program for the secondary schools. At this time G. D. Birkhoff made a significant modification of Euclid's axioms, following the general pattern set by Hilbert, but effecting a great economy by using the order and completeness properties of the set of real numbers. The problem which Birkhoff hoped to solve in this way was aptly described, in a similar attempt by Saunders MacLane, as one of finding "a simple and intuitive set of facts on distance and angle which suffice to characterize plane geometry."[4] Under National Science Foundation sponsorship, the Birkhoff axioms, modified in turn by Edwin E. Moise, have been used to prepare experimental textbooks[5] which are being tried in many schools throughout the United States.

Another secondary school approach to the study of geometry, initiated in Germany and used there today rather extensively, is suggestive of some of the aspects of Klein's "Erlangen programm." The group of transformations—rotations, reflections, and translations—are used to characterize Euclidean geometry, but are introduced by a system of axioms preserving Euclid's congruence of triangles as fundamental to the further development of the study of geometry. Bachmann[6] has shown that this group can be replaced by the group of reflections alone.

The survival of Euclid's geometry rests primarily on the assumption that it is the only subject available and suitable for initiating young minds into the nature of a mathematical axiomatic structure. In rebuttal of this notion, it should be noted that for well over a half century, we have had developments in arithmetic and in algebra which exhibit in an elementary manner the nature of axiomatic structure, yet at both the secondary and the college level, until quite recently, we have avoided teaching these subjects in any other than their classical form. It is now evident that if we care to do so, we can use branches of mathematics other than geometry to develop the notions of axiomatic structure, and thus permit a treatment of school geometry from what may be called a contemporary view-

[4] Saunders MacLane, "Metric Postulates for Plane Geometry," American Mathematical Monthly, Vol. 66, pp. 543-555, 1959.

[5] School Mathematics Study Group, Geometry, Vol. I and II, Yale University Press, New Haven, Conn., 1961.

[6] Bachmann, Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer-Verlag, 1955.

point. Despite all the knowledge at our command, the present reformed treatments of geometry in the United States are aimed essentially at preserving Euclid by introducing the real numbers to repair the flaws. The conferences at Aarhus (1960) and Bologna (1961) were centered largely on presenting systems of axioms that would preserve Euclid. Our progress at the secondary school level seems to have become stagnant in the Euclidean pool.

In the last few years in the United States, there have been some indications of a strong desire for change in the geometry offered at the collegiate level. "College geometry" has almost disappeared. The geometers of the modern variety have "long since removed all these Euclidean treasures to the museum where the dust of history has quickly dimmed their luster." Similarly, analytic geometry, as a separate subject, is fast disappearing, and the essentials of this discipline now appear in the study of analysis or of the calculus. In place of these geometries, a trend that has been gathering strength involves the presentation of courses that examine the "Foundations of Geometry." In such courses, the monumental work of Veblen and Young on Projective Geometry has been serving as a source book. Another book, more limited in scope, which is enjoying a revival of interest is the work of H. G. Forder on The Foundations of Euclidean Geometry. This treatise gives a complete structure of euclidean metric space, making use of axioms selected from those given by Hilbert, Peano, Pieri, Huntington, and others. A book that brings into the undergraduate program frontier thinking in mathematics is a work by L. M. Blumenthal called A Modern View of Geometry. Starting with the concept of ternary ring developed by Marshall Hall in 1943, Blumenthal provides for a system of coordinates for the affine plane and eventually develops a set of metric postulates for the euclidean plane. Finally, we have the treatises by Levi[7] and Libois,[8] both of which approach the study of geometry via affine space.

If one were asked to give one outstanding characteristic of twentieth century mathematics, it would most likely be its unity. This is largely the result of the all-pervasive use of sets and structures. A significant development in geometry, uniting it more closely with mathematics as a whole, is the treatment of space by vectors. This movement can be described as one in which mathematicians, looking at vectors as conceived and used by physicists, have transformed them into a mathematical structure for the study of

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[7] Levi, Howard, Foundations of Geometry and Trigonometry, Prentice-Hall, New York, 1960.

[8] Libois, Paul, Introduction á la Géométrie, Chap. I, II, III, University of Brussels Press, 1960.

space. The gradual evolution from equivalent classes of free vectors in a plane or three-space, to centered vectors, to a vector as an ordered set of only two or three real numbers, to arithmetic vector n-space, to metric and topological space, is the great modern achievement in geometry.

Attempts to introduce vectors into the development of analytic geometry have made some headway in European countries but have met with no success in the United States. A textbook by Reynolds written in 1928, and another by Murnaghan in 1946 never gained entry into collegiate study. Within the last few years this treatment has been making an appearance in textbooks on calculus under the heading "vector space." In this context, the purpose of the inclusion of the topic is to develop those portions of affine and euclidean two- and three-space necessary for the application of the differential calculus to the study of curves and surfaces. Even where these texts are used, however, the presentation of vector space has not been noticeably successful. There is strong reason to believe that the difficulty lies in the lack of preparation on the part of the students to study space in any other manner than through Euclid's synthetic development.

REFORM IN INSTRUCTION

In what follows, I shall speak of Euclid's geometry, meaning thereby the synthetic presentation of the Elements of Euclid, with any modern refinement such as that made by Hilbert, Forder, Birkhoff, or others. I shall also use the term euclidean geometry to mean a space conceived as a set of elements called points, into which there has been introduced a structure and a metric which gives the distance-preserving relations that one finds in Euclid's geometry. This euclidean space is at the heart of mathematics, and its properties furnish the means for extending, and generalizing many of the other branches of mathematics, e.g., relativity, differential geometry, groups, metric spaces, and topology. Euclid's geometry, on the other hand, has nothing to do with these subjects; it is now sterile, off the main road of mathematical advance, and can be safely relegated to the archives for the use of tomorrow's historians.

The way of looking at geometry, or indeed any branch of mathematics, is quite different today than it was at the beginning of this century. At that time there was a set of objects (usually assumed to exist and not described) upon which was superimposed a total structure in the form of a complete set of axioms. (One need only refer to Hilbert and his Foundations or to Huntington and his twenty-seven postulates of complex algebra for samples of this interpretation.) The subject was then complete except for the tautological development of the theorems in the structure.

Today we start with a basic set, the elements of which are either undefined or frequently constructed from more fundamental sets. In geometry we refer to the set as a "space" and to its elements as "points." We then introduce a structure on the set, and develop those properties possible under that structure, e.g., an affine vector space. On this structure, in quite different ways, we can introduce further structure, thereby obtaining a new space; e.g., introducing a norm in the affine vector space gives us a normed vector space. We can now extend the structure further by introducing a metric (a distance function) into our normed space. Depending on the function defining distance, we get different spaces. If one uses the euclidean function, we obtain euclidean vector space. Depending on the description of our "points," our space can have 1, 2, 3, n, or infinite dimensionality.

If the method described above of using an ordered pair (set, structure) to define a mathematical system is in keeping with the spirit of the times, and if the permissibility of introducing various structures on a set gives flexibility to the system, then we should capture some of this spirit in the study of high school geometry. We need some bold new thinking in geometry that dares to tear us away from Euclid. This was the thesis of Dieudonné at the Royaumont conference, where he indicated lines along which the reform could be carried out. Using his remarks and the thoughts of others who participated in the Dubrovnik seminar, I would like to propose the following as a program of instruction in geometry from age 11 years up to the collegiate years of study. Time permits the presentation of only the major outline of the program, but it is relatively easy to fill in the details.

HIGH SCHOOL GEOMETRY

Before entering the secondary school at age 11 years plus, the child has amassed a large number of geometrical ideas, all physical in nature. Capitalizing on this knowledge and using laboratory techniques of measuring, folding, drawing, and making of models, the whole body of information in Euclid's elements of plane and solid geometry can be acquired and used during the ages 12 to 15 years. During this period, little by little, the essential conceptual elements, such as point, figure, line, plane, space, must be abstracted as purely mental constructs, and relations between these elements generalized, to the extent that small deductive chains of theorems can be established on less than an axiomatic base.

During the ages 14 to 15 years the student will have had further deductive work in algebra as he studies new number systems and algebraic structure.

At ages 15 to 16 years the student should be ready to combine algebra and geometry in a study of affine plane geometry. The following sequence is merely suggestive of what can be done.

1. The introduction of equivalence classes of free vectors; the sum of two vectors (as an operation different from that on numbers); the group of vectors under addition; theorems and exercises.

2. The product of a vector by a real number or scalar multiplication and its properties. These properties will permit the proof of all theorems of affine plane geometry. The important properties are:

(a) Given a point P and vector $\vec{V}$, a line is the set of points M for which $\overrightarrow{PM} = r\vec{V}$

(b) $\vec{V} = k\vec{V}'$ implies V and $\vec{V}$ are on parallel lines, and conversely

(c) If $k\vec{V} = h\vec{V}'$ and $\vec{V}$ and $\vec{V}'$ are not on parallel lines, then $k = h = 0$

(d) If $a\vec{V} + b\vec{W} = c\vec{V} + d\vec{W}$, then $a = c$, and $b = d$

3. The introduction of coordinates and centered vectors; the basis of the coordinate system: parallel projection and the components of a vector. Now we can establish the equation of a straight line in the affine plane and study its properties.

4. The introduction of perpendicularity and inner product. Define $\vec{a} \perp \vec{b}$ if $\vec{a} + \vec{b} = \vec{a} - \vec{b}$. We can now develop the Pythagorean theorem, the law of cosines, and all of Euclidean plane geometry. In addition, we can tie algebra to geometry by giving a vector solution to a pair of linear equations in two unknowns.

All this work is real geometry—elementary, and preparatory to the way geometry is used in physics and in analytic geometry. It can serve as an illustration of an axiomatic development. It possesses structure. It follows the suggestions which were made by both Dieudonné and Choquet at Royaumont and Aarhus, respectively, and it is in full agreement with the view expressed by Henri Cartan at Bologna, when he urged that the student be led to the study of affine geometry and vector spaces as directly as possible. In his advocacy of this method, Cartan did not intend to exclude any pedagogical steps necessary to make possible the transition from intuitive geometry to this material. He did urge that all unnecessary subjects, no matter how seductive (projective geometry, finite geometries, etc.), be deferred.

At ages 17 and 18 years this geometry can be further extended by introducing euclidean space, first of one, two, and three dimensions and then extended to n-dimensions. Along with the study of sets, set operations, and mappings, and the use of coordinate systems, this is the best preparation in geometry for the study of analysis that we can have in this day and age.

At the collegiate level, a systematic study of vector spaces and linear transformations (linear algebra) can complete this study of geometry. At this stage, one would hardly be inclined to question whether he is doing algebra or geometry. In addition, there should be a study of pure geometry from the modern viewpoint, treating the foundations and relations of finite geometries, affine geometry, projective geometry, conformal geometry, the nature of space (metric, topological, and transformations).

To summarize: I have tried to point out a number of major movements of mathematical thought that should influence our attitude toward instruction in geometry. These are:

1. The discovery of non-euclidean geometries and its implication for axiomatization of all branches of mathematical study.

2. The classifications of geometries by Riemann, Klein, and others.

3. The arithmetization of mathematics. This was done, as we know, by Dedekind, Weierstrass, Cantor, and others.

4. The development of differential geometry.

5. The perfection of Euclid's geometry at the close of the last century.

6. The development of vector, metric, and topological spaces.

7. The trend toward mathematical structures and mathematical unity.

8. The importance of all these movements for the recognition of what is possible and what is desirable as geometrical instruction at the secondary school level. This indicates:

   (a) The present treatment of Euclid's geometry must go. It contributes little to further study and is outside the mainstream of mathematics.

(b) Euclidean space is important and must be at the center of geometrical instruction. It must be developed as arithmetic space, with a vector structure, and euclidean metric.

(c) All of Euclid's plane and solid geometry must be learned informally in the junior high school.

(d) In the emerging college programs in analysis, an important part of the study consists of vector spaces and linear algebra. The high schools have the responsibility of preparing students to look at space from this point of view.

To accomplish all this we need pioneering and experimentation for the purpose of producing a high school program in close harmony with the contemporary spirit of mathematics, and one that will permit flexibility to adaptation to newer procedures in mathematics as they emerge. We must never again permit one geometry so to dominate the program of instruction and the minds of men that it bars any change, and this is exactly what Euclid has done over the last one hundred years.

* * * * *

Discussion Following Address by Professor Fehr

The vector approach for secondary school geometry proposed by Professor Fehr met with strong approval as well as equally strong opposition.

Professor Luis A. Santalo of Argentina urged the construction of a common syllabus of material to be taught regardless of the axiomatic system used as a basis—Euclid, Hilbert, or a vector type.

In reply to this suggestion, Professor Fehr reiterated his stand against keeping Euclid's synthetic approach, claiming that it "no longer has the value it used to have, because the step to dimensional varieties cannot be based on it." He also said that "axiomatics" are out of place at the secondary level, and that some topics of the traditional course would have to be omitted.

Part of a statement by Professor Stone on the matter of "cutting" is quoted here. Professor Stone said: "In summary, the ultimate decision will have to be made by the teacher or a group of teachers, without forgetting that a proper balance must be maintained because in teaching too much emphasis along a given line could go against the diverse interest of students. Let us remember, for example, the tremendous importance given some fifty years ago to conic sections.

Obviously it is necessary today to replace a large amount of the material formerly used, but in any event we must avoid committing the mistake of the past, that is, stressing a given line too much."

To a question as to whether there was information on experimental secondary school programs in a vector form of geometry, Professor Fehr referred to an experiment in a small number of New York City schools, and to one conducted by Mademoiselle Félix of the Lycée Fontaine in Paris. He also suggested School Mathematics Study Group materials on vectors as a good source, but admitted that the books for secondary schools were still to be written.

In arguing for the retention of Euclid, Professor Coleman of Canada suggested that everyone interested in mathematics found his first incentive in Euclid himself. "If, on the other hand," Professor Coleman said, "we tried to do things axiomatically, we would have to admit that the problems derived from an axiomatic system would look almost trivial compared to those developed over two thousand years of Euclidean teachings. Therefore, if the time previously dedicated to Euclid were used now to develop a modern axiomatic system, we would have to develop sufficiently complicated and interesting problems for these new systems."

Professor Fehr's reply to this statement was that there are problems in the study of vector spaces that are as interesting as those found in the traditional Euclidean geometry, or even more so.

With Professor Pauli's reference to a successful program in Switzerland, based on a study of vector spaces and going back ten years, the discussion period came to an end.

*THE TRAINING OF TEACHERS OF MATHEMATICS**

Professor A. Valeiras and Professor Luis A. Santaló (Argentina)

The problem of the training of teachers of mathematics presents diverse aspects which arise from the following considerations: (1) General considerations pertaining to any level and any type of school. (2) Special considerations depending upon the level of education and the type of school. (3) Considerations involving problems unique for a particular country. With reference to group (2), in which an analysis is made of the differences between teachers of elementary, secondary, and higher levels of teaching, we shall limit ourselves essentially to the training of teachers of the secondary schools. As to group (3), taking into consideration the objectives of this conference, we shall refer to certain characteristics common to all Latin-American countries instead of limiting ourselves to any one country.

GENERAL CONSIDERATIONS

Every teacher must have an adequate knowledge of:

1. What is taught, that is, a clear and extensive knowledge of mathematics.
2. How to teach, that is, a knowledge of didactics.
3. Whom he is teaching, that is, a knowledge of the individual he is teaching: psychology of the student.
4. Why and for what reason he is teaching, that is, a knowledge of the general problems of education, such as purpose and instruments of education, development of character and personality, relation between school and society, systems of schooling in the various countries and in different epochs.

The emphasis which must be placed upon these four factors in the training of teachers depends upon: (a) The level of teaching. (b) The historical moment in which we live. Let us examine these two items separately.

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*Address translated from the Spanish. Address delivered by Professor Santalò.

The Level of Teaching

For the elementary school (pupils age 10-12 years), the items of greatest importance are items 2 and 3, that is, didactics and child psychology. The first item, what to teach, is also important, but except for details, there is general agreement on the material to be taught, which is therefore somewhat standardized.

For the university or higher levels of instruction, the first of the items is the one to be stressed. That a university teacher should know mathematics is of utmost importance. And what is more, he should, if it is at all possible, be actively engaged in mathematical research, which is the only way to acquire a knowledge of the subject.

The real difficulty lies with the secondary school, where, at first sight, the four items seem to present the same "coefficient of importance."

The Historical Moment in Which We Live

The rhythm of evolution of every society must be felt by the teaching of any discipline at the moment it is presented. This is particularly true in the case of mathematics, where content as well as method depends largely upon the actual state in which the evolution of science finds itself, since science always finds support in mathematics or in the nature of mathematical reasoning.

In times of slow scientific progress, the mathematical needs of each of the activities to which the student might dedicate himself in the future are well known. Relatively stable programs can be prepared and the teachers can receive a training adapted to those programs which most probably will require few modifications during a long period. "What" to teach is well known and, for this reason, there is time to intensify "how" to do it, to give attention to the psychology of the student, to consider relations between the school and society, thereby achieving a more perfect, integrated education.

On the contrary, in times like ours, in which scientific knowledge extends itself impetuously in depth and breadth, with a speed that makes noticeable the changes during the period (from twenty to thirty years) in which a teacher performs his mission, the mathematical preparation of the teachers not only cannot be static and definitive, but must possess the necessary excellence and training to adapt itself to the possible changes and evolve with their rhythm. The first item, that is, the teacher's ability as a mathematician, is, to the best of our knowledge, the most important.

Today it frequently happens that many mathematical facts and skills, once considered superfluous or unnecessary for certain objectives, suddenly begin to have a use in vital technical problems or applied sciences, or are considered to be indispensable within mathematics itself. The teacher of mathematics must be prepared to assimilate new mathematics and be up to date with these novelties.

If it were possible to define for each teacher a "coefficient of knowledge" equal to the ratio between the mathematics he knows and the mathematics he teaches--a coefficient that in many instances not so long ago, and in some instances even now, was and is just about the unit--we would say that the main objective on which to concentrate any effort of reform in the training of teachers should be to increase this coefficient to a maximum.

TEACHING IN THE SECONDARY SCHOOL

The secondary school receives children who are about 12 years of age, and, when its cycle has been completed, discharges young people 17 to 18 years of age. During this period, the student has completed a stage of his adolescence involving the physical problems of fast growth, and the appearance of sexual and psychological functions, problems deriving from the fact that these functions presuppose a change from childhood to adulthood and the encounter with all the difficulties which such a change implies. The school must accept the psychological instability that accompanies adolescence and must have a rational structure which permits it to adapt itself to its constant evolution.

A fundamental change has taken place in the teaching at the secondary level during this century. At the beginning, this type of teaching was received by only a minority of youngsters, generally members of well-to-do families with a high level of education. Today, on the contrary, the majority of adolescents attend the secondary school, or at least intend to do so. Economic reasons no longer limit attendance, in general, because of the facilities which are offered. On the other hand, the greater knowledge demanded by the complex and more technical modern society disposes parents and children to regard this educational period as a necessity. The increase in the secondary school population has been a reason for not observing, in the majority of cases, minimum requirements for a teaching position. This situation has two different aspects: one, the lack of adequate buildings to house the large number of students, and the other, the lack of sufficiently prepared teachers to cover the ever increasing demands for teaching personnel. The multiplication of secondary schools was made on the basis of improvisation in both these aspects but the structure of the schools has remained unchanged. These schools have almost completely escaped all influence from the tremendous advance registered by all the sciences, or by the development of their applications. We continue to teach in the same way we taught thirty years ago, in spite of the fact that during these years many ideas have become obsolete and notable efforts have been made to synthesize knowledge. Making a massive operation of teaching has worked against its own evolution.

There has been, generally speaking, no opportunity for experimental schools, and the teachers continue to work under the same conditions that existed when they were students. Even when maintaining the same quantity of effort or the same initiative, the terrific growth of the mass has worked against its speed of evolution and improvement.

To give some details about this situation and to understand the essential points requiring modification, it is advisable to analyze the actual status of the secondary school teachers in the Latin-American countries.

Generally speaking, this group of teachers is composed of: (a) Graduates from establishments especially created to train teachers for the secondary school: institutes for teachers, higher schools for teachers, pedagogical institutes. (b) University graduates, with additional preparation in education. (c) University graduates, without additional preparation in education. (d) Teachers of the elementary schools or graduates from elementary schools without any other degree. (e) Persons without any degree or title.

The teachers of groups (a) and (b) have had training for a period of three or four years; this gives them the technical capacity to teach and covers two aspects, the educational and the specifically mathematical. The courses in education include psychology, philosophy, pedagogy, history, methodology, and administration, which are completed with some time given to practice teaching. The courses in mathematics include algebra, geometry, and trigonometry, with the subjects of the secondary programs, in addition to analytic geometry, projective and descriptive geometry, infinitesimal analysis, and some physics.

The preparation described above corresponds, generally, to the most favorable instances of specialized mathematical studies. When the teacher is prepared to lecture in other sciences, the mathematical studies are reduced.

The teachers in group (b) who attended some regular courses in mathematics at a university before they received their training for teaching (for example, those corresponding to a licentiate in mathematical sciences) have a deeper mathematical preparation. They have taken some courses in advanced mathematics (modern algebra, analytic functions, functional analysis).

The number of teachers in these two categories is a minimum;

rather than the rule, they are practically the exception. Because of the great need for teachers and also because of the discretional methods commonly used in making the appointments, a large number of teachers in categories (c), (d), and (e) have been given positions which they still hold.

In the case of group (c), which includes, for example, engineers, architects, land surveyors, chemists, while their scientific preparation might be adequate, they have had no training in education, and frequently have no interest in the educational process, which precludes any possibility of efficacy in their teaching.

In the other two groups, there is a lack of scientific preparation which is essential to the fulfillment of the mission of teaching mathematics. It is curious that while in many places the necessity for avoiding quackery, when it concerns the body, has become an established fact, lack of preparation is tolerated and even protected when it concerns the mind, when the truth is that the damage caused can be as severe and as irreparable with the second as with the first.

To obtain appointments in teaching, there are different procedures. We mention some of them: (a) By competition of record and written examination. (b) By competition of record only. (c) Automatically by the authorities at the end of the period of schooling. (d) Without any type of competition or selection. (e) By combining (a) and (d), depending on the teacher's having or not having a degree signifying competence. (f) By the local authorities of the zone of the school or the principal of the school, by examination of record together with personal interviews.

For private schools, where the exclusive interest of the principal is the smooth functioning and prestige of the school, method (f) can be both advantageous and efficient. For public schools, where interest in the promotion of education is lost among many adverse factors, almost always foreign to the school, the only methods which offer sufficient guarantee are (a) and (b). Nevertheless, method (d) is still widely used, placing the teachers at the mercy of political changes, personalities, and interests which have nothing to do with the function of the educational activities.

In the Latin-American countries, the system of payment according to number of hours of lectures is used. The appointed teacher has 3, 4, 6, or even 24, 30, or more hours of lectures and the salary paid is a linear function of this number. The only requirement for the teacher is adherence to this timetable in giving the corresponding lectures. This system favors the consideration of teaching as a means of securing an income, without engendering any special interest for the task that is being done. It also creates the very common situation in which a teacher has to lecture in several schools to maintain himself economically, and also provides the schools with many teachers none of whom can live on that income only. A change to a system of permanent teachers for each school, with full-time appointments, is indispensable if we wish to create conditions conducive to improvement in the efficacy of education.

The economic conditions which face the majority of teachers in the whole world--not only in America--are not satisfactory. The poor social position of teachers, especially in large cities, is in great part a consequence of the precarious economic situation provided by the teaching profession. It would be useless to talk about providing the teacher with an excellent preparation for him to pass on to his students, if on the basis of that preparation, industry and commerce can offer a better salary and greater possibilities of advancement.

Another condition that is necessary to the well-being of the teacher is definite encouragement and reward for effort. Uniformity in financial arrangements for teachers is very common; their income is determined only by the number of hours of lecturing. Teachers with excellent preparation and a desire for advancement have the same income as those with insufficient preparation and no desire for improvement. This is not the best way to encourage the first group and cure the indifference of the others. Improvements, if any at all, come only with seniority in the position, and in an automatic and uniform manner, for all members of a staff. It should be the responsibility of the principal of each secondary school to identify those who work with enthusiasm and dedication, and obtain for them benefits which offer encouragement for further improvement and reward for work well done.

Setting aside economic, professional, and social problems, consider the scientific conditions in which the teacher of the secondary school has to work. The situation in this instance is not a bit more attractive. The preparation in special schools for teachers does not put the student in contact with the most advanced centers of learning of the country, although it can provide them with very good teachers. On the other hand, on graduation, the daily work in the classroom is a contributing factor in withholding from them the possibility of bringing their knowledge up to date, a possibility which is completely void for those who have to teach in towns far from centers of study.

POSSIBLE REFORMS

Suggestions for removing the impediments noted above follow.

Complete dedication of time to position and adequate salary. All tasks well done demand exclusive dedication to them. The teacher of mathematics should be completely absorbed in the problems inherent in his profession. In particular, if conditions require a type of teaching that is flexible and adaptable to continually changing programs and new standards of education, full-time work in the position is an essential.

The routine teacher who devotes an hour every two days to the repetition of theorems in solid geometry, and is absorbed in some other professional activity the rest of the time, must pass into the archives of the history of education.

What is more, the full-time teacher must have a number of class hours that will allow him time for reflection and study in connection with the technical problems of mathematics and the educational problems of his profession.

Finally, fair remuneration is also an essential. Assuming that an excellent program for the training of teachers provides them with a good mathematical background, if present conditions of low and insecure salaries persist, we take the risk of having those teachers leave the profession for work in other better paying positions in industry or technical organizations.

University training. The programs for the training of teachers of mathematics must be planned, in their technical aspects, within each country and by the best mathematicians. Only they can decide which topics are indispensable in the subject areas that must be studied by future teachers, and in what depth they should be taught. Since the best mathematicians are in the universities, the training of mathematics teachers must be assigned to these centers of study.

Naturally, mathematical knowledge does not account for the whole picture. It is necessary to complement it with courses in didactics, psychology, and pedagogy. At this moment, however, these complements to the indispensable are still complements, in the sense that they would be useless without the necessary mathematical preparation. We can select the approach already taken by many European countries and in some countries of this continent, of complementing the studies of the licentiate in mathematics (studied at a university) with studies of about one year in a special teacher-training institution. This plan of study would lead to a degree in teaching, and the individual who has successfully completed such a program would be qualified to teach in a secondary school.

The complementary year of study has the advantage of making distinctions in the preparation of teachers according to the type of school in which the teaching will be done: lyceum, technical school, or academic school. For the last, it would be useful for the teacher to have contact, for a period of several months, with commercial and industrial enterprises so that he can better understand the environment in which his students will work.

Periodical in-service courses. A strong initial mathematical preparation is an important factor if the teacher is to be able to keep up to date in the mathematical field. But in any case, it is always advisable to help him improve on his background. For teachers in active service, who were never given instruction in modern mathematics, and for those who teach in places remote from higher centers of culture, this help is indispensable.

This is why summer courses or courses organized for vacation periods are commendable. In many countries, this type of program is already in existence. In the United States of America, the National Science Foundation sponsors many such courses and the influence which these exert is extended to Latin-American teachers of secondary school mathematics through the scholarships that the Foundation has been granting to them. In Argentina, the first of a series of courses organized by the National Council of Scientific and Technical Research will be given, even though there are others being planned by the Faculties of Pure Science. In Brazil, the Institute of Education, Science, and Culture, through its São Paulo branch, is planning to develop a series of mathematical programs, extending those already in operation for the other sciences. The same type of activity is going on in other countries.

Publication of syllabi and pilot texts. We have said that it was our belief that teacher of mathematics at the present time should have an extensive and profound knowledge of mathematics, even at the expense of other types of knowledge which are very important but which occupy a secondary place in the changing world of today. It is obvious that this comment is rather ambiguous and as such suffers from the same defect that characterizes the majority of proposals and recommendations for change. In reality, the only way to be clearly understood is to prepare a detailed syllabus, with an outline of topics, programs, and the length of time to be devoted to each item. And it would be even clearer if to this syllabus was added a textbook adapted to it.

In this respect, we believe it would be important to sponsor a group of experts to draw up a pilot curriculum with an outline for each subject and a bibliography best adapted for the training of teachers of the secondary school. It might be impossible to get these experts to agree on criteria, in which case it would be possible to work out several curricula and programs. But always, however, there would emerge a common denominator to be used as a basis for future reforms in each country and for each type of school.

Less spectacular, but nevertheless just as important, would be the publication by well-known experts of lists of topics and subjects that, in their opinion, should be included in the in-service courses mentioned above. While it is true that the selection of these subjects can change greatly from one country to another, nevertheless, in the case of countries in Latin America, the differences are not very noticeable. This suggestion would involve not only an outlining by topic but, in a most specific manner, a statement of what subjects of modern mathematics the secondary school teacher should know, aside from those he has to teach, in order to do his job of teaching with greater efficiency.

The "general culture" of the teacher of the secondary school. With the additional material that increased mathematical knowledge will force us to include in any new program for the training of secondary school teachers of mathematics, it is necessary to point out certain omissions and eliminations that will inevitably result from an attempt to provide time for the new subjects without lengthening the period of study. We believe that these must be made in the subjects labeled "general culture" in many programs for the training of teachers.

Nobody will dispute the fact that teachers of the secondary school should possess a broad background of general culture. The debatable point lies in deciding whether this general culture should be included in the regular courses for the training of the teachers or whether it should be acquired by the teacher himself after he has received a foundation in such courses in his period of secondary schooling.

It is evident that a good teacher of mathematics must have a general knowledge of literature, philosophy, and history, but he does not have to be an expert in any one of these subjects. Is it justifiable to include in his curriculum courses in these subjects that because of their breadth cannot be treated other than in a most superficial manner and will decrease the time available for a good mathematical training?

It is taken for granted that because of his position in society, a teacher of mathematics must have some knowledge of, for example, certain trends in painting, the existence of entities like UNESCO, and Churchill's role in World War II. But, is it necessary to include painting, world organization, or contemporary history in the curriculum of the teacher of mathematics? What a good teacher needs above all else is to know mathematics—the more the better. With this basic preparation, having awakened within himself a curiosity about knowledge through his own reading, through attendance at conferences, and through discussions among colleagues, he will acquire general culture.

Let us make it quite clear that in this phrase "general culture" we do not mean to include those studies to which we have referred to above—psychology, didactics, and pedagogy—which must be studied as a specific body of knowledge, preferably after the mathematical studies have been completed and always from a scientific point of view.

*THE PREPARATION OF TEACHERS OF MATHEMATICS**

Professor Omar Catunda (Brazil)

In preparing this address, I must confess that on several occasions I was almost overwhelmed by what seemed to be a task beyond my powers.

There were several reasons for this feeling. In the first place, although a thorough reform of the teaching of elementary mathematics is under consideration, a decision on what this reform should be, even in the more basic subjects, has not been reached, and we have to face the problem of preparing a teaching staff who will have to work with a reform which will soon be in full development despite the fact that what preparation this staff will need is not yet known. In the second place, we, the professors of the Department of Mathematics of the Faculty of Sciences of the University of São Paulo, feel that the instruction given in this faculty, although acknowledged generally to be the best in the country for the preparation of scientists, is far from being satisfactory insofar as the preparation of future teachers is concerned. Up to the present time, either because of the scarcity of teaching personnel or because of the rigidity of political leaders, we have not been able to resolve this difficulty. Finally, the gap between the actual situation in the field of education in my country and, if not the ideal, at least a satisfactory situation is so great that the perspective of what needs to be done to improve the situation before it deteriorates even more produces vertigo. The sensation is one that is experienced by those who have the task of reconstructing a ruined edifice whose foundations are constantly in need of shoring during the progress of the work.

It is because of the immensity of the task that I take the liberty of digressing a little to discuss a problem that is somewhat related to the theme of this address. I refer to the novel ideas on the teaching of elementary mathematics. The structure of teaching in my country would with difficulty support a radical reform even of the most rational type possible. In particular, the premature introduction of algebra, it seems to me, would be disastrous, since one of the greatest faults of the secondary school in Brazil is its formalism, that is, the emphasis which teachers place on definitions, rules,

*Address translated from the Spanish.

and formulae which the student must memorize, with tremendous loss through neglect of the development of the powers of reasoning. It is my opinion that as long as the student is not capable of reasoning in connection with concrete facts, such as problems concerning money, fractions, proportional numbers, interest, he should not be subjected to the formalism of algebra. However, I do agree that when it is introduced, it should be treated in a more modern fashion, that is, it should be based on the fundamental notions of sets and operations on systems of elements which might be numbers, translations, symmetries, etc.

Another feature in which Brazil differs completely from the situation to be found in Europe concerns Euclidean geometry. Professor Dieudonné claims that secondary education loses too much time with classical geometry in the manner of Euclid. In Brazil, the problem is different. With the liberty that is accorded teachers to teach a mere 75 per cent of the syllabus (a percentage which can be reduced considerably), we frequently find students who have had practically no geometry. And since this is the only part of elementary mathematics that cannot be taught except through reasoning (other parts can be replaced by a number of formulae and rules learned by rote), what I would proclaim for Brazil would not be "Down with Euclid!" but "At least Euclid!"

The training of teachers of the secondary schools of Brazil is the task of the Faculties of Science, official, or private with official authorization. These are higher schools, sometimes separated from other schools of the University, where, therefore, the professors for the most part are supposed to be engaged in research. Actually, this is true only in a small proportion of cases. The majority of the professors of mathematics of the Faculty of Science limit themselves to giving courses that are expository, often repeating a textbook or notes, and their activities after classes consist of one or two conferences, special courses, and the like. Some teach in more than one of these higher schools, and there are others who teach in their free time to add to their professional income. It should be noted that while it is necessary to submit to competitive examinations to obtain a professorship in the official (public) schools, this is not the case for the private schools, although they are nevertheless licensed by the government.

In spite of the fact that the law requires certification by the Faculty of Science, only a small proportion of the teachers in the secondary schools meet this requirement (16 per cent in 1958 and 20 per cent in 1961). Because of the small number of teachers available for the large number of schools, always increasing, many years after the graduation of the first students certified by the Faculty, in 1936 and 1937, the administration, by means of Ministerial decrees, authorized the ce.tification of teachers without specialized preparation. These in many instances were appointed in a most arbitrary fashion, and later were allowed to keep their provisional positions by submitting to a very simple examination, called a "sufficiency" examination. Even so, the scarcity of personnel was so acute that teachers who failed this examination were permitted to keep their posts.

To obtain a permanent position, that is, a "chair," the government provides a "competition for admission" examination in which newly graduated licentiates and teachers who only have certificates are permitted to participate (the latter were allowed to participate for the last time in 1961; only those with diplomas are now permitted to try for a permanent post). The law still permits the licentiates in physics, pedagogy, and social sciences to take part in this examination, since a portion of the program for these students has some concern with mathematics education, though very small in the case of the last two.

What I have said can be corroborated by the report made for this conference by my colleague, Alfredo Pereira Gómez.[1]

With these considerations in mind, it can be seen that actually the influence of the faculties and, in particular, of good faculties, on teaching in the secondary schools is still very slight and probably will continue to be slight for many years to come. Without doubt, there is prestige associated with the best faculties, where are to be found the most famous mathematics professors of the country. This prestige is, however, more in the nature of morale and I think it will be several years before it will have any influence in the administration of the schools.

It is my belief that one of the outstanding factors in the sad state of affairs in Brazilian education is the low level of esteem, if I may be permitted this expression, accorded teachers. This is apparent when one examines the student body that enters the Faculties of Philosophy and Mathematics. It is a fact acknowledged by those who are examiners that the intellectual level of engineering students, even only in so far as mathematics is concerned, is considerably higher, as a rule, than that of students who enter the section of the Faculty of Mathematics. It is also possible to observe that the students in the secondary schools who show an exceptional disposition toward mathematics (and this quality rarely is to be found with conformity to a modest scale of living) for the most part desire to enter the school of engineering and at present, thanks to the success of the last years in the study of the atom, they often

[1] Editor's note: The material of this report has been included in the statistics given for the countries participating in the Conference and presented in Part IV.

choose the courses in physics in the Faculty of Science. This year I have been teaching a group in which at least seven of the students can be classified as the best I have encountered in twenty-seven years of teaching on the Faculty of São Paulo, and all of them are physics students.

The motive is evident: on the one hand, there is a career of an engineer, with an initial salary about five times the minimum legal salary, with good prospects of success with large companies, in construction, in industrial enterprises, etc., even with the possibility of accumulating a fortune. On the other hand, consider the career of a teacher, with an initial salary two or three times as large as the minimum if the teacher is limited to 12 classes a week (compulsory minimum), and possibilities of increasing it to even eight or nine times that salary, but at the expense of the quality of his teaching, which will certainly be attenuated until he becomes merely a teaching machine. This being the case, it is possible to find teachers who lecture in ten or eleven classes a day. It is obvious that such teachers cannot fulfill their obligations. I believe that a secondary school teacher, to be really effective, should allow three hours of work for each hour of lecture. These hours should be used for preparation of material of the lecture, preparation and correction of tests and school exercises, conferences with other teachers and parents of the students, and time for self-improvement and study. The possibilities open today for careers in research, statistics, or other applications of mathematics are numerically so unimpressive that they do not constitute an element of attraction for young students.

Throughout this exposition, I have been concerned not with the teaching of elementary mathematics for those who will require that knowledge in their profession, i.e., mathematicians, physicians, engineers, and technical personnel, but secondary teaching as a contribution to the training of character, the development of the power of logical reasoning, and as a basis for the study of any other science. It is for the majority of students who complete secondary school that I observe with great sadness the fact that the teaching they receive not only is lacking in quality but is progressively deteriorating because of scarcity of personnel, poor administration, absence of integrity in promotions, and similar reasons. I will not take time for anecdotes concerning the points I have just made, which, moreover, are known to exist in every country; nevertheless, I have personally seen, for example, few students in their last years of secondary school who could explain why a term of an equality changes sign when it passes from one member to the other. If among the students of our own Faculty, 80 per cent of the students of a particular class are capable of solving a second degree inequality because they have learned the rule for obtaining a solution, at least 80 per cent were perplexed, during an examination, because the unknown was replaced by its absolute value. In another examination, only two out of a class of 50 students could show that the interior of a convex figure in ordinary space is also a convex figure.

The very somber picture that I have painted does not mean that I am pessimistic about the future of my country. Seeds have been planted there that, in due time, will be able to produce very favorable reactions which will bring about an amelioration of the condition, although it may take several years.

Undoubtedly, among the factors that guarantee the possibility of improvement is the increase in number and quality of research mathematicians in the best universities and institutes of mathematics. In the years 1957, 1959, and 1961, these mathematicians convened for the Brazilian Colloquia on Mathematics, where they met with many young students and foreign guests. During the three colloquia, which had an attendance numbering 50, 70, and 120 persons respectively, mathematics was not the sole subject under discussion. Problems in the teaching of secondary school mathematics were also considered.

Moreover, secondary school teachers have been meeting every two years since 1955 in a Congress on the Teaching of Mathematics. The topics under discussion at these congresses have been the reform of programs of instruction, new methods of teaching by groups, under the guidance of an assistant or a student, and so on. Other meetings called "teachers' meetings" have taken place for the interchange of ideas. Nevertheless, this movement is still at a beginning stage and, as a high-ranking officer in the Ministry of Education put it recently (in referring to the ideas proposed and adopted almost unanimously in the Congress of 1955), six years is a short time to influence the bureaucratic machinery in obtaining alterations in arrangements that are in force.

In the last few years, the world movement in favor of a reform in the teaching of mathematics has had strong repercussions; nevertheless, mathematicians and teachers in Brazil have not yet met to discuss this matter in an orderly fashion. Meanwhile, several schools have been experimenting under special authorization from the Ministry of Education. A highlight of this work was the visit of Professor George Springer to São Paulo this year. During August and September, Professor Springer conducted courses for secondary school teachers on "Mathematical Logic," organized seminars, and at the same time, gave a course on the "Theory of Sets" and one on "Modern Algebra." In the last two undertakings, he was assisted by members of the Department of Mathematics of the Faculty of

Philosophy of the University of São Paulo. The courses were made possible through the combined efforts of the National Science Foundation at Mackenzie University (São Paulo) and were also a result of the visit by Professor Oswaldo Sangiorgi, of the University of São Paulo, to the United States, where he took a course at the University of Kansas under a scholarship grant from the Pan American Union. We can hope for best results from the courses of Professor Springer, since all the teachers who took these courses were dedicated to the study of the experiments taking place in the United States and are now, on their own, repeating the same experiments.

Let us now consider the real theme of this address. I have said before that the preparation of teachers is under the jurisdiction of the Faculties of Science. The curriculum of these faculties, official or authorized, is set by the Ministry of Education, as is the system of promotion for the students. Within the existing curriculum, there is ample room for the establishment of a good program, and in some recently created faculties, the young teachers have taken this opportunity to give quite a modern organization to the teaching of mathematics.

A few years ago, at São Paulo, we adopted a course of study in which the subjects were grouped into sequences. Students were permitted to register for courses in the various groups without attention to sequence except for subjects where prerequisite courses were required. The following is a listing of the different groups of subjects offered at São Paulo, together with indications of the time allotted to each.

- Projective and Analytic Geometry—2 years.
- Survey of Mathematics (Concepts of Modern Algebra and Theory of Equations)—1 year.
- Mathematical Analysis (Differential and Integral Calculus with one or more variables, numerical series, functions, power series, elementary differential equations, analytic functions)—3 years.
- Survey of Geometry (This is the name currently used to designate Modern Algebra—linear algebra, finite groups, rings, fields)—1 year.
- Advanced Analysis (General Topology, Lebesque Integration, Hilbert Spaces)—2 years.
- Advanced Geometry (Galois Theory, Algebraic Geometry, and possibly Differential Geometry)—2 years.
- Foundations of Mathematics (Theory of Numbers)—2 years.
- Experimental and General Physics—2 years.
- Rational and Analytic Mechanics—2 years.
- Mathematical Physics (Trigonometric Series, Partial Differential Equations, Special Functions)—1 year.

The course was planned for four years, but a large proportion of the students take a longer time to complete it. The students who pass all these subjects receive the degree of bachelor. Any one who completes, in addition to the above, courses in Educational Psychology, General Didactics, and Special Didactics (with practice teaching) receives the degree of licentiate, which will, from now on, be required of all who take the competitive examination for a teaching position in the secondary schools.

The course is taught on a very severe basis and at times becomes quite heavy; but it offers an excellent foundation, in many respects, for those who intend to dedicate themselves to mathematical research, to teaching at a higher level or in other faculties, such as schools of engineering. There is, however, a general feeling that a reform is needed, and the problem will be up for discussion next year during the vacation period.

During the last few years, there has been considerable discussion concerning the curriculum of the Faculty of Mathematics, with a view to simplifying the course for teachers of mathematics. Although several proposals have been made, not one has been accepted in its entirety for reasons of tradition and organization. There is general agreement, however, on the following two points:

1. It is desirable to maintain a common curriculum, at least in the basic subjects, for future research mathematicians and future secondary school teachers.

2. The complete course for teachers of the secondary school should be given by the Faculty of Science and should be as strenuous as any course of university level.

From among the proposals made during the last few years, one by Professor Alexandre Martine Rodriguez Filho deserves special mention. In addressing the Second Brazilian Colloquium on Mathematics and the Third Brazilian Congress on Mathematical Education, Professor Rodriguez advanced a thesis that had its origin in the observation that the Faculties of Philosophy have not in practice changed their structures for twenty-five years, while the social structure of the country has greatly changed. Thus, whereas previously the student body was drawn from the more favored classes, today a large group of the students come from the laboring classes with a result that there has been a tremendous increase in all the Faculties as well as in private and public faculties in the small cities of the interior of the country. The proposal of Professor Rodriguez is that the program in the Faculties of Philosophy be divided into two periods, the first of three years' duration and the second of two, and that the division be in accordance with the following plan:

1. Two years of common basic subjects, for mathematicians and teachers alike.
2. One year of elective subjects in which future mathematicians would select courses in specialized mathematics and the candidates for teaching positions (on the secondary school level) would be advised to select courses of a more elementary level, such as History of Mathematics, Metric Spaces, Elementary Theory of Groups, Differential Geometry, in addition to courses in pedagogy.

The students who completed these three years of work, including the courses in pedagogy, would be licensed to teach in the secondary schools. The last two years would be dedicated to the training of scientists; the work would be very intensive and all the students would be given scholarships.

At the Third Congress on Mathematical Education mentioned above, Professors José Manuel de Cruz Valente and Elon Lages Lima also made suggestions. The latter is also co-author with Leopoldo Nachbin of "La Enseñanza de las Matemáticas en las Facultades de Ciencias y en las Escuelas de Ingeniería del Brasil,"[1] which has been published in the bulletin of the Sociedad Paranaense de Matemática.[2] In this article, there is a proposal for a reorganization of the course of study with respect to the simplification of instruction in the departments of mathematics and in the basic course for teachers and research mathematicians. It also includes a proposal for the creation of courses in Applied Mathematics, including the Calculus of Probability and Numerical Calculus. It is also suggested that the student take only four subjects each year, choosing during the first year, for example, either Physics or Applied Mathematics. During the fourth year, the students would be separated into two groups, one consisting of those who elect a scientific career requiring special courses and the other, future teachers taking work in the field of pedagogy.

Giving due consideration to the proposals to which I have made reference, revolving them in my mind and studying them in the light of my personal experiences, I would go further. In the present state of teaching, since we have to make up for an accumulation of lost time it is not enough for students to have a university training. The less able students can, I have noticed, learn the subjects of advanced mathematics and can even complete and pass their examinations, and then encounter great difficulty with elementary material. This

[1] "The Teaching of Mathematics in the Faculties of Science and in the Schools of Engineering of Brazil." [2] Mathematical Society of Parana.

can be verified principally in connection with Rational Arithmetic and Synthetic Geometry. Thus, it is necessary for the Faculties, without impairing the teaching of basic material, to conduct a general review of elementary mathematics. This could be done with the creation of special courses for future teachers. At São Paulo, Professor Benedito Castrucci developed a course a few years ago in Non-Euclidean Geometry, designed especially for future teachers who can take this course as an elective in the fourth year. I myself have taught, among others, a course on a subject unknown to the students, Spherical Geometry. On other occasions, I have taught courses in seminar which involved a complete discussion of problems of elementary mathematics, particularly of geometry. These subjects have also been taught several times in vacation courses, given by the Faculty for secondary school teachers.

I wish to call attention to another difficulty which we have not been able to resolve. Teachers of physics and mechanics, in order to describe phenomena and discuss problems, often have to make references to concepts of analysis, such as, for example, integrals, integrals of surfaces and volumes, differential equations, before it is possible to present these topics in the mathematics courses. According to these teachers, the solution to the problem, which I consider to be very poor, is to explain important results briefly and to have the students accept their word that the rules given are true and will be explained at a later date in the course on Analysis. This problem will also be a topic for discussion during the next vacation period.

Having presented, with sincerity and objectivity, the various problems involved in the training of teachers, I do not feel that I can offer a completely detailed solution. Nevertheless, using the thesis of Professors Alexandre Rodriguez, Lages Lima, and Leopoldo Nachbin, I believe that because of the present conditions in Brazil, I must support the following proposals:

1. The training of teachers should cover a period of three or four years of university work.

2. The first years should be the same for future teachers and future mathematicians as far as basic subjects are concerned.

3. In these years when the courses are common to both groups, there should be no attempt to reach a profound and abstract level of mathematical theory. Indeed, an intensive course should be given--with many applications and exercises--in the basic subjects, such as algebra, analytic geometry, differential and integral calculus, as well as such subjects as analysis, analytic functions, linear algebra, differential geometry, which give a panoramic view of modern mathematics without demanding great efforts in preparation for them.

4. Courses and seminars on reviews of elementary mathematics should be established for future teachers.

5. The addition of applied mathematics is important, but there would have to be agreement on the possibility of excusing the student of mathematics from the courses in physics and mechanics if he substitutes for courses in the latter subjects work in applied mathematics, as has been suggested by Professors Lima and Nachbin.

6. It is necessary to maintain constant contact between the Faculty and the teachers in the field with a view to permitting those who desire to do so to improve their level of knowledge. This contact may be maintained not only through courses in the vacation period, but also through conferences and more frequent discussions between teachers in the higher schools and those at the secondary level.

To facilitate the last proposal, it would be helpful to establish a periodical, as I have frequently proposed; but times are not yet propitious for such a project with any guarantee of continuity of publication.

In closing, let me thank you for your attention and for the opportunity given me by this Inter-American Conference on Mathematical Education in the expectation that this meeting may serve to clarify doubts and questions that I have raised and that, on my own, I have not been able to answer.

* * * * *

Discussion Following Addresses by Professors Santaló and Catunda

The addresses by Professors Santaló (Argentina) and Catunda (Brazil) concerning the preparation and training of teachers of mathematics promoted a lively discussion in which several delegates expressed dissatisfaction with the state of mathematics education in their respective countries. None of the participants in the discussion took issue with the speakers and no questions were raised in connection with any of the suggestions for the improvement of mathematics education presented in the addresses. The remarks, with some omissions, are reproduced below (in translation) because they are representative of the sentiments of the group and provided a background for the resolutions that were later adopted by the Conference.

Prof. Balanzat (Venezuela): It is my belief that teachers of mathematics should be trained in a university, and, furthermore, that the abandonment of European practices in this matter is the cause of the low quality of secondary school teaching about which we have been lamenting. Nevertheless, if the Conference should make recommendations in keeping with the conviction expressed above, it would be practically impossible to realize them. However, we might solve this problem by recommending that teacher-training institutions provide a program of the same or almost the same level as that of the Faculty of Mathematics [of the University].

Prof. Alfaro (Costa Rica): I should like to point out to Professor Catunda that if he thinks the situation in Brazil is bad, in my country it is catastrophic. The University dates from 1942. In 1957 the Institute of Physics and Mathematics was founded and has just this year graduated its first student. From 1942, when free education was established, to the present time, the number of secondary schools has grown from 8 to 58. You can imagine how great a number of teachers had to be provided in a hurry; elementary school teachers who had to be recruited to teach in the secondary schools possessed only the background in mathematics that they needed for their degree....This year a law was passed requiring that teachers permanently appointed to the secondary schools have a degree in the subject they will teach. I wonder how this law will be enforced when there are not enough candidates who meet these requirements.

Prof. Babini (Argentina): If we are to introduce topics and problems from modern mathematics, it is absolutely essential that teachers know them perfectly. I believe this to be the central theme of our discussion. Mastery of the new subject matter can be achieved through special courses adapted to our needs.

Prof. Santaló (Argentina): I believe the teacher of secondary school mathematics should be trained in that institution where the best mathematicians of the country are to be found, whatever name this institution bears. This does not mean that I advocate the closing of those establishments which are now functioning, but I would place them in direct contact with others where, for one reason or another, the mathematicians enjoying the greatest prestige in the country are to be found.

Prof. Alfaro (Costa Rica): The problem revolves around those teachers who are now in service and whose collaboration is indispensable. These must, in a short time, attain the same level of knowledge as the newly graduated teachers. To accomplish this, in Costa Rica, we have instituted a system of special courses during the vacation periods. The result has been that teachers who could pass these courses have not been compensated for the extra training they received, while those who could not pass lost prestige. The outcome has been that the courses have been poorly attended. I believe that a recommendation should be forthcoming from this conference to the effect that our governments should be urged to adopt measures to compensate those teachers who submit to the extra training, or to require attendance on the part of all.

Prof. Garriga (Puerto Rico): There is the problem of explaining to teachers why they should give up their old methods. Our courses in the capital have been planned for this purpose. Our experiences with these courses have been varied. Some of our teacher improve rapidly and with enthusiasm. However, when they return to their schools, they find that their places have been taken by those who remained behind.

Prof. Catunda (Brazil): In Brazil, we have a system that assigns points to those teachers who attend courses of this type. These points can help them at a later date to get better schools. However, the success of the program is limited. If twenty sign up for one of these vacation courses, ten attend, five take the final examination, and two pass. There is, however, official recognition in Brazil for the training obtained by the teacher in these courses.

Prof. Marcelo Santaló (Mexico): ... We also have summer courses; attendance is free of charge and at the end there is an award of a certificate of attendance that can be used in future competitions. I wish to point out that the participants in these courses do not submit to any examinations and are therefore not subjected to the risk of failure. I believe that a possible solution to these problems would be to allow teachers a sabbatical year every five or six years for the purpose of giving them a chance to put themselves in contact with the mainstream of present-day mathematics. The Mexican Mathematical Society, a private organization, has sponsore congresses and conferences in the States at which university professors have given lectures on modern mathematics.

A recommendation that could be made at this conference is that leading mathematicians of each country regularly give some of their time to the exposition of modern mathematics in articles and in reviews or periodicals which should be established for this purpose.

Prof. Laguardia (Uruguay): ... As for promotions, in my country, at any rate, there is a grave problem. The teacher starts his career in a certain position, and, with the passage of time, his hours decrease and his pay increases. This is quite just, but it becomes an obstacle in that the only way to better one's position is by achieving seniority in the job. I believe that an outcome of this conference should be a recommendation to the effect that teachers with initiative should step up to higher positions in the scale without having to wait their turn. Naturally this program would be tied in closely with an expansion in each country of the inspectorate.

I believe another fault in present-day teaching of mathematics in all spheres is that teachers put more emphasis on teaching facts than on developing mathematical power.

Prof. Choquet (France): I believe the greatest deficiency in secondary school teachers lies in the fact that they do not have a sufficiently solid background in mathematics. When a teacher does not have complete mastery of the subject he teaches, his definitions and proofs are developed with subtle errors. I believe that one of the most important questions is that of determining the length of time which must be employed in the training of the teacher. It would be useless to shorten the period, because if there were a greater number of teachers, these would be mediocre and their teaching would deteriorate little by little. As for the training in pedagogy that is necessary, the teacher could complete and perfect it by reading articles, reviews, etc. Nor should his cultural training be neglected. The teacher should know how to write correctly and to explain with clarity and precision any subject whatever. Experience tells me this is not generally the case.

Prof. Pereira Gómez (Brazil): The candidates for teaching positions in mathematics at the secondary school level are not, at least in my country, the best and most capable of the young people with an inclination toward mathematics. The reason is the bad situation in which these teachers find themselves. When a young person feels inclined toward mathematics, he almost always selects a career such as physicist, engineer, in preparation for which, he will study a great amount of mathematics. This type of career, furthermore, will raise him to an advantageous social and economic situation. The position of teacher does not even provide inducements of a professional nature, since the young teacher will be drawn into the vicious circle of the type of teaching created ad hoc by the poor instruction that is usual for our countries.

Prof. Hernandez (Nicaragua): I propose that an Inter-American Center of Education, similar to other organizations such as the Inter-American Center of Housing, be created. Problems confronting the countries of South America are not the same for all; the situation changes radically as one goes from Brazil to Costa Rica. In Nicaragua, we do not have a Faculty of Pure Mathematics; this discipline is, for the most part, in the hands of engineers.

Moreover, the government of Nicaragua, and this is generally the case for all countries of Central America, is faced by all sorts of financial problems. For this reason, we have decided to take the initiative, in an extra-governmental manner, to create a university having in its title the descriptive words "Central-American" and in which the course in mathematics of the Faculty of Engineering has been increased from the usual two years to three, thereby permitting the introduction of subjects which up to this time have been absent from the curriculum—subjects such as numerical analysis and matrix calcuius.

Prof. Laguardia (Uruguay): I would like to present some results of a statistical survey conducted among 600 teachers in the public secondary schools of Chile. Of this number, 40 per cent are properl licensed, 30 per cent have been trained by normal school teachers or by accountants, and the others have studied for one or two years at the university.

Prof. Sevilla (Honduras): I believe that, to be effective, the recommendations issuing from this conference must be completely general. We have arrived at the conclusion that the secondary schoo teacher should be trained in centers where the best mathematicians of the country are to be found. I believe that we should give concrete suggestions for the rapid training of teachers for the present emergency. Moreover, we must decide whether it is more practical to raise the level of training of the teachers at present on the job or to replace them gradually by new and better prepared personnel.

Prof. Catunda (Brazil): In Brazil we have chosen to increase the size of the faculty and to insist that all new appointees be licensed. I believe, however, that we will have to study the possibility of resolving the problem by emergency courses for teachers now in service.

Prof. Laguardia (Uruguay): The problem might be resolved, at least partially, by the expedient of denying promotion to any teacher who has not demonstrated his interest in improving his training and in extending his knowledge.

Prof. Coleman (Canada): The situation in Canada is reassuring Beginning teachers can obtain salaries even higher than those given beginning engineers. This is the outcome of the work of the Institute for the Development of Mathematics. Teachers unions have obtained this advantage with threats of strikes so that, in the last ten years, salaries have been doubled. Actually a teacher with ten or twelve years of experience can command a salary of about $9300 (U.S. dollars).

Prof. Robinson (West Indies): Our profession is as worthy as any other and merits the same treatment. One of the recommendations to be made by this conference should be that teachers be accorded a high rank in the social scale and receive greater remuneration. It is advisable that future teachers should be instructed by the best mathematicians since good teachers will train good teachers.

Prof. Arias (Guatemala): I believe we have to make an emergency suggestion for improving the qualifications of teachers in active service. For this purpose, a system of correspondence courses might be tried.

Prof. Babini (Argentina): The Higher Council of Research and Engineering has conducted by means of a system of scholarships, special courses for teachers in active service. The courses are taught by university professors. The teachers of the capital have created an association in the form of a club, without the usual objectives of a union, to provide for contacts among the members, and to furnish, in a special way, an atmosphere for discussion and debate on scientific and pedagogical problems.

Prof. Catunda (Brazil): In São Paulo we have recently created a Center for the Teaching of Mathematics where courses have been conducted with excellent results.

Prof. Garcia (Puerto Rico): With the help of the Faculty of Science we have been developing summer institutes since 1957. There are also other types of courses for teachers who are employed during the day; for example, evening classes for the purpose of extending the mathematical knowledge and horizons of the teacher. We also have a system of academic-year institutes to which teachers are sent for a year of study under the most distinguished professors of our universities.

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The Round-Table Discussion

The interchange of views and experiences reported above served as groundwork for a round-table discussion. Professor Rafael Laguardia (Uruguay), the moderator, opened the session with a short talk outlining the common problems in mathematics education confronting the countries of Latin America. Professor Laguardia's remarks on this occasion are reproduced below.

THE TEACHING OF MATHEMATICS IN LATIN AMERICA

Professor Rafael Laguardia (Uruguay)

It is seldom that we mathematicians and teachers of mathematics of the entire continent have occasion to gather together, and in the company of eminent colleagues from other lands, to deliberate on prospects for the development of our science in our respective countries. We must therefore take advantage of the opportunity that this conference provides to exchange ideas on the direction which the teaching of mathematics should follow, to survey the problems that are unique for Latin America, and, in particular, to consider the obstacles that stand in the way of the full realization of our objectives and by what means they can be removed.

A glance at the reports that have been submitted on the state of the teaching of mathematics in our various countries will show that, granting differences in degree of scientific progress, the development of this teaching is hampered by similar circumstances not to be found in the more highly developed countries of other regions. Let me enumerate some of these difficulties:

1. In spite of compulsory attendance at the primary level, a large proportion of children of school age do not get any schooling, or leave school prematurely. The percentage of dropouts at the secondary level is even higher.

2. The efficiency of education is impaired at all levels by serious defects in organization. Teachers are often forced to reduce the time they devote to teaching in order to eke out a living. Full-time employment of a university faculty, the rule in Europe and North America, is the exception for us, and is frequently misunderstood to mean only a full-time load of teaching.

3. In a great many cases, there is lack of the necessary coordination among institutions, such as university schools of science, teachers colleges, and research institutes.

This enumeration could go on to take up most of our available time. Since we may expect this to be the first of several meetings of this nature, I have thought it wiser to stress the following problems, the solutions to which are particularly urgent, and to restrict the discussion to these topics:

1. Raising the social and professional status of the secondary school teacher of mathematics.

2. The same problem at the university level, especially in so far as it concerns mathematical competence.

3. Effects on education of the population explosion, and emergency measures required at the secondary school level to meet this problem.

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In addition to Professor Laguardia (Uruguay), the moderator, the panel consisted of the following members: Professors B. Alfaro (Costa Rica), J. Babini (Argentina), P. Casas (Colombia), A. Pereira Gómez (Brazil), L. Schwartz (France), M. Stone (U.S.A.), J. Tola (Peru), and G. Torres (Mexico).

The organization of the teaching of mathematics at the secondary school and university levels furnished the theme of the panel discussion and was considered in relation to the following topics: Remuneration of teachers. Teaching obligations. Tenure. Mathematical preparation of teachers. Diplomas required of teachers. Emergency measures for meeting demands for teachers caused by the population explosion. Official evaluation of students' work. Time allotted to the study of mathematics, with reference to the adoption of new programs.

In response to a question about television teaching of mathematics, Professor Carlos Federici (Colombia) gave an account of a course in mathematics that was taught by television in Bogotá.

The substance of the conclusions reached during the round-table deliberations was embodied in the set of resolutions which were drawn up at the close of this session. These resolutions, in the form in which they were finally adopted by the Conference, are presented in Part III of this report.

*THE NEW MATHEMATICS AND TEACHING**

Professor Gustave Choquet (France)

We have six days to investigate a vastly extensive field; moreover, we represent different countries, all with varying needs. Our task, therefore, appears to be an onerous one; in reality, however, it is from its vastness that we will draw our ultimate success. We will have to eliminate everything that is not essential, put aside all details of our diverse national problems, and go directly to the common substratum.

The subject of this paper is the teaching of mathematics. As in most discourses on instruction, we shall concern ourselves with the student, the teacher, and anything that happens between them. If one of these three links is missing, the chain is broken. We shall think of this chain as a system of mutual interchanges between the student and the teacher.

I shall limit myself to the mathematical aspects of the problem, but it should not be forgotten that the personal attitude of the teacher is more important than the subject matter taught. The important feature is that the training of the mind and the desire to learn and to be creative should remain with the student after instruction will have been given. The young mind can be killed just as effectively through the medium of modern mathematics as it can with mathematics from another era. At the end of his period of schooling, the student should have learned that mathematics is the creation of man and that new mathematics can be manufactured in the same way that a new pair of shoes can be made; and he himself, at his own level, should have felt the exaltation that accompanies creative pursuits, in creating, even though on a very modest scale, his own theorems and his own solutions to problems.

The favorable climate in which these exchanges, student $\rightleftarrows$ teacher, occur should not be lost from sight. We should note that in all mathematical activity there are four stages: observation, mathematization, deduction, application. Instruction should not be limited to the deductive stage. Mankind through the centuries has created new concepts which are expressed in words, in systems of definitions. The discovery of these is as important as that of a new tool,

*Address translated from the French.

or of a mutation affecting our body. In the former case, it is a question of a mutation (not apparent) of the brain which happens after birth with the help of a catalyzing agent which in this instance bears the name "teacher." The goal of teaching is the transmission of mutations from an organism already changed to one that is still young and supple.

These mutations involve a reconstruction of the brain; they cause the passage from one level of thinking to a higher level. The process most favorable to these mutations is the study and observation of mathematical situations chosen for their suitability to this task by the teacher.

MODERN MATHEMATICS

I wish, for the moment, to act as your conscience, to analyze, in your presence, facts which you all know very well, and to try to deduce from this analysis the directions in which we should employ our efforts to harmonize our teaching with the prodigious development of science in our century.

AXIOMATICS AND FUNDAMENTAL STRUCTURES

The history of mathematics has brought to light many efforts at synthesis: the axiomatization of Euclid, the creation of algebra and of its early notation, the synthesis of Descartes. Our era is characterized by the explosive nature of this effort toward synthesis and of its effects—an explosion born in the creation of the concept and language of the algebra of sets.

This means an ever greater and more sweeping unity. There is only one Mathematics. One of the factors for unity is to be found in the method of axiomatics: the separation of the main sources of reasoning, the isolated study of each of them with a view to the ultimate re-combining of the constituent elements.

These main sources are the structures. For example, on the real line R, we have the structures of group, field, vector space, of order, of topological space, etc.

A structure is defined over an arbitrary set by axioms (as, for example, axioms of order).

First there are the mother-structures, which we find in all theories, such as relations of equivalence, structures of order, algebraic structures, topological structures. Multiple structures (for example, a topological group) involve two or more mother-structures tied together by axioms of compatibility. Cross-structures which involve many mother-structures (for example, the theory of potential) and which are of interest to a great variety of mathematicians, constitute the domain of Analysis.

In this way, algebra takes on the appearance of a study of a world made up of crystals (structures defined by a small number of axioms), and analysis becomes a study of complex parts. All this makes for a luxuriant jungle which can be explored in many ways.

THE NATURE OF THE AXIOMATIC METHOD

1. Economy of thought. Taylorization: mother-structures are the machine tools. Economy of concepts and of notations; a choice of precise wording, which is suggestive, and promotes resonances. It has led to a greater clarity in writing, and to a new style.

2. Multivalence in the theorems studied (in contrast to the univalence of the axiomatics of Euclid and to that of Peano). This multivalence guarantees its adaptation to varying situations: the difficulty of deciding that certain statements have to do with algebra, or geometry, or with analysis (elementary geometry = linear algebra on R^3; quadratic forms = the study of conics; the geometry of Hilbert space is a convenient translation of the theorems of analysis). This multivalence is therefore a factor operating for unity, allowing for a mutual enrichment of diverse theories.

3. Utilization of the "context." An entity is no longer studied by itself; we now study large classes of entities from which we obtain generalizations and, at the same time, a mutual clarification (real-analytic functions are better known by studying them in the domain of the complex, "normal families" of analytic functions, etc.).

Modern mathematics is therefore interrelated and in it an internal "dynamism" is revealed in vocabulary and typography:

$$x \to f(x); \quad A \to \overline{A}; \quad x \sim y; \quad A \times B; \quad E/R$$

The entities are transformed continuously in the hands of the modern mathematician. Only relations, which require dynamic exploration, are of interest. The virtuosity of young people trained in this school of thought guarantees that the method is well adapted to the human brain.

4. The greater adaptability of mathematics to the physical world. The more multivalent an axiomatic system is the greater the number of cases it encompasses; on the other hand, there is a very slight chance that a univalent system would be a good schema for the physical world (frame for general relativity theory furnished by Riemann spaces).

5. Process of validating obscure and metaphysical concepts. A good system of axiomatics has often been the only means of terminating endless discussions of a metaphysical nature. Example:

complex numbers; and, more recently, the calculus of probabilities, clarified and made more fruitful by means of Kolmogorov's axiomatics.

Dangers of the Axiomatic Method. A Warning

A machine tool can operate in an empty space and manufacture nothing of interest.

Criterion: Do not use "a brick to kill a fly." A general theory is not justified unless it can reveal hidden and fruitful ties among theories which up to that time had been seemingly unrelated, or unless it can produce a solution for a problem which had not yet been resolved.

There is need of a protective device which will guard one from the temptation of developing an axiomatic system as an end in itself.

These constitute the great problems of abstraction. "A branch of science is alive only if it offers an abundance of problems; a lack of problems is a sign of death." (Hilbert)

METHODS OF DISCOVERY LINKED TO AXIOMATICS

Certain methods of discovery find meaning only in the study of multivalent structures. The following are some methods which the serious researcher will rediscover elsewhere by himself.

1. Method of weakening the axioms. An analyst tries to prove a theorem E in a cross-structure S ; this theorem would have a meaning in a system S' which was less rich in axioms than S and which would simplify his study. If the analyst proves E for system S', E is at the same time proved for S. If he can, in S', construct a counter-example C which would show the theorem E to be false, a detailed study of C could lead him to formulate a supplementary property P which, adjoined to S', would provide for the possibility of proving E. He would then have to go back to S to see if P is true in S.

2. Method of strengthening the axioms. We adjoin new axioms to the system S, i.e., we study particular cases. Thanks to the new axioms, we are better able to prove P; we return then and try to adapt to S the results obtained. (Example: the study of Markov processes with discrete or finite sets.)

3. Study of neighboring-structures. If it is impossible to prove a property P in a structure S , we try to prove it in S' , a neighboring structure. It might happen that a part of the reasoning might be transferrable to S and we can try to modify slightly the theorem P to make it true for S. (Example: the attempts to prove the

Riemann hypothesis by first studying it in connection with finite fields.)

Such an attempt might lead to posing the problem in a more general frame in which the hints for attacking it are more easily seen.

THE NEW FACE OF MODERN MATHEMATICS

The use of axiomatics and the language of sets has given a new look to the developments in mathematics of the last thirty years.

1. Multivalence and generality. There is a predilection for multivalent structures, and for general theorems ("When it doesn't cost more, we develop a theory in its most general frame." Bourbaki). This requires economy of thought but also greater efforts at abstraction from the reader.

2. Invasion of all of mathematics by algebra. The algebra of groups, sets, linear and multilinear algebras, duality. A taste for properties that can be expressed in algebraic form.

What was at one time "Analysis" has found its algebraic equivalent (R replaced by a field of characteristic 0 or even any field whatsoever; derivative is defined on an algebra A as a linear mapping D of A into A so that $D(xy) = x\,D(y) + D(x)\,y$).

3. Process of evolving definitions. This process is particularly interesting to observe in Bourbaki; it is one of the aspects of the "bourbakization" of theories: When it is evident that the only part of a concept A that is used is a property B, then this property is turned into an axiom that may or may not be equivalent to A; from this one gains in generality and, in all cases, in clarity and convenience. Examples: (1) The theorem of F. Riesz establishes the equivalence of the Stieltjes integral on R and linear positive forms on $K(R)$ from which is obtained Radon's well-known definition on measure. There is a gain in generality from this because we can replace R by any locally compact space, and also a gain in flexibility (operations on measures, topology).

In fact, an even greater gain is obtained since the chosen definition is an open door for the admission of a crowd of analogous definitions: the generalized surfaces of L. C. Young; currents of de Rham; Schwartz distributions; and, more generally, continuous linear forms on a topological vector space.

(2) Measures invariant on a group: Natural extension of the Lebesque measure on R defined as a positive linear form on $K(R)$, invariant under translations of R.

(3) Definition of measurable functions on $[0,1]$, for example, taking as a definition the classical theorem of Lusin.

4. Choice of subject matter and of definitions. The danger of constructing vast theories, but vicious ones, is so great that the mathematician must select with extreme care the object of his study and his definitions. He must choose tools which have proved to be useful in diverse domains. This is true for an even stronger reason if it concerns teaching. It will then be necessary to banish elegant results and even profound ones, if they present only an impasse or a conclusion of a theory. It will be necessary to eliminate, without care for completeness, concepts close to those judged most fundamental (because if the mind is divided it loses its power). In elementary questions, for example, emphasize the great tools: characteristic properties of R (totally ordered commutative field, for which every segment that is bounded has an upper bound), the theorem on finite increments under a form valid in R^n, convex functions, etc.

IMPLICATIONS FOR TEACHING

The traditional mathematician studied particular entities which he grouped according to their coloration (arithmetic, algebra, geometry, trigonometry, etc.). The discovery of the big structures has changed the warp and woof of the fabric of our world. In place of horizontal fibers, we see vertical ones.

Such a revolution cannot remain cloistered in the domain of research. It is necessary to take a new look at teaching at all levels, primary, secondary, technical, university, in light of the discovery of the great structures.

We will see an ever-increasing unity in mathematics and a greater unity in the teaching of the subject at all levels. The slogan will be: "Algebra and the fundamental structures from kindergarten to the university!"

No teaching will be possible if we do not clear out our domain; without the processes which bring with them an economy in thinking, we overwhelm our children under a mass of inheritances of the past. Instruction based on the historical method, for example, has become unthinkable. It is necessary to clear the ground.

First, as for the future of mathematics: It is neither the old men nor even the mature ones who are the inventors of mathematics; it is the young people. In order to make it possible for them to assimilate mathematics, the great simplifying concepts must be clarified for them and they must be taught theories which unify and connect the various disciplines.

Second, as for the users of mathematics: They should have access to the simplifications made possible by modern mathematics. True, there is a scarcity of books for teaching it. It is the urgent duty of mathematicians to write these books and to aid their colleagues in secondary, technical, and higher schools in writing them.

As for those who will not become mathematicians nor users of mathematics: The study of the theory of sets in connection with the study of elementary logic and of some multivalent structures is good training and is entrancing.

Although in the days that follow, we shall have occasion to return to the discussion of instruction in detail, it would be well at this moment to consider certain principles.

1. We should accustom our students to think in terms of sets and operations as early as possible. It will be necessary to teach the simple, universal, and precise language of sets. At the same time we should teach them the rudiments of logic in its relation to the grammatical study of their language (to negate a proposition, to understand the force of the words <u>and</u>, <u>or</u>, <u>for all</u>, <u>there exists</u>).

2. At a very early age, our students should have a clear understanding of the concept of function. They should be able to construct various examples of functions in arithmetic, algebra, physics, and to produce the composition of two functions, to take the inverse function of a biunique function, to recognize a group of transformations.

3. The students should be able to recognize at an early age the relation of equivalence (numerous examples; quotient-sets), relations of order, and they should study some concepts of topology.

4. In all fields, it will be necessary to get directly at the essential tools that have numerous and immediate applications.

For example, in arithmetic, we should eliminate various rules accumulated through the centuries, for the sake of inserting some simple algebra as a replacement.

We should define properly, but not metaphysically, what we mean by two proportional numerical functions ($f = ag$), instead of making vague references to "proportional magnitudes."

In geometry, we should renounce completely all worn-out methods based on "the case of equality of triangles," and bring out, in contrast, the vector structure of the plane and of space, furnished with a scalar product.

We should also be able to give up the jungle of theorems of metric geometry which have been accumulating down through the centuries. It will be profitable to study some of them as exercises; but their study should not be permitted to prejudice the study of the tool subjects. The set R of real numbers is an essential one of these. At all stages, advancing in precision with the age of the student, we should bring out the characteristic properties of R: that it is a commutative field, totally ordered, continuous (with initial "segments" bounded from above). To attempt to construct such a field would, on the contrary, be useless (in any event, before the university level).

However, we are already starting to give details, but this is the work we have set for ourselves for the next few days, when we shall be studying how we can adapt to the various levels of instruction the principles on which we will have reached agreement.

SOME CHARACTERISTIC TENDENCIES IN MODERN MATHEMATICS

Professor Marshall H. Stone (U.S.A.)

As mathematicians we are all more or less aware of what a prodigious development our science has undergone since the beginning of the Renaissance—and more specially since the closing decades of the nineteenth century. We are only slightly less aware of the spreading confidence in mathematics as one of the intellectual tools indispensable to every kind of human endeavor and speculation. The catalogue of those arts and disciplines in which mathematics is called on to play a leading or at least a decisively important role no longer begins and ends with physics, chemistry, and engineering. It is growing year by year and already includes an impressive variety of special fields in both the biological and the social sciences. Nor does this extension of the power of mathematics rest exclusively upon a more skillful and refined exploitation of the classical techniques developed by the great mathematicians of the past—Euclid, Archimedes, Descartes, Leibnitz, Newton, the Bernoullis, Euler, Laplace, and Gauss, to mention only some of the most illustrious masters of our beloved science.

We know very well that the day is long since past when physics and chemistry could be understood or even put to practical uses without a good command of many newer parts of mathematics, in particular those which treat of groups, vectors, tensors, transformations, matrices, and the like. To put this in another way, classical geometry, the calculus and the theory of differential equations despite their amazing effectiveness in serving the end for which they were first developed—namely, to interpret nature in terms of the concepts made precise in classical mechanics—have been found inadequate as soon as it became necessary to introduce new concepts and to think in terms of relativity and of quantum theory.

So far as the biological and social sciences are concerned, many of the early attempts to emulate the successes achieved in physics by using the classical mathematical tools met with frustration. As a result, the current tendency, specially in the social sciences, is to employ different and often quite unrelated techniques, sometimes drawn from various parts of modern mathematics, sometimes devised upon the spot for the particular task in hand. Enough has already been accomplished in certain directions by the use of game theory and the theory of linear programming—both very new chapters in mathematics—so that this tendency has been significantly reinforced. In short, we live in a period when we can neither remain unaware of the changing role of mathematics in the evolution of science as a whole nor permit ourselves to ignore the need for fully exploiting all its resources, new as well as old, in an intensified effort to broaden our scientific understanding of the universe and to apply that understanding to practical human affairs.

As teachers of mathematics we are all beginning to recognize that the almost explosive proliferation of mathematics and its applications during the present century has inevitably brought us face to face with an extremely critical situation in the field of mathematical education. At the same time we are all acutely conscious of the fact that in dealing with this situation we are compelled to deal also with another extremely critical problem on a strictly practical level—the problem of recruiting and training enough teachers of mathematics to meet the demands of our younger generation. This younger generation is too numerous and too much interested in mathematics for us to teach it adequately without increasing our ranks and even our average competence as teachers of mathematics. In looking for ways in which it may be possible to attack these difficult and challenging problems, we are discovering that one of our principal tasks is to elaborate new mathematical curricula for schools, universities, and normal schools. The purpose of doing so is to incorporate into our courses of instruction, at the appropriate places and on the appropriate levels, those newer parts of mathematics which are the most important for our students to know and understand. At the same time we need to eliminate some traditional topics which have become obsolete or obsolescent because of our mathematical progress. The task is one which we cannot regard as satisfactorily completed until the new mathematics courses have been tried and tested. By implication, therefore, this task involves the writing of texts, the preparation of teaching materials, and the study of teaching methods which promise to stimulate our students' interest and ease their progress through a richer but more demanding program of mathematics courses. It is evident that to achieve all this we must make a substantial capital investment in education and must organize our knowledge and our energies in a highly efficient manner.

Nevertheless, the essence of the problem is to decide the shape which the new mathematics programs should take if our students, of whatever individual inclinations, are to be adequately prepared for careers in a world where science and technology are certain to exert an ever more dominating influence. Everything else flows from this central decision; and none of the other educational problems involved can be taken up intelligently before this decision has been made. What, then, should be our guide in designing the new curricula needed to bring the teaching of mathematics up to the level of our times? A little reflection suffices to show us that we can find such a guide only in a comprehensive philosophical grasp of the nature of modern mathematics and its role in the scientific process. Thus, before we begin our study of the mathematics curriculum, we must address ourselves to an inquiry into these most fundamental matters.

In many ways one of the most prominent trends in modern mathematics is the increasing importance assumed by algebra and algebraic methods not only in the various branches of pure mathematics but also in the most diverse applications of mathematics. At the same time algebra itself has undergone a very extensive development, broadening its scope and enriching its content. Indeed it is just this internal development which has given algebra the flexibility and versatility enabling it to enter so intimately into other domains of mathematical and scientific thought. Thus one of the principal objectives of curricular reform must be to introduce modern algebra into our elementary mathematics program at the school and university levels. While something has already been done in this direction in some of our universities during the last few decades, very little indeed has yet been accomplished in the secondary schools. Most of what is offered today as modernization of the secondary school courses in algebra is nothing of the sort, as it fails completely to convey the spirit of modern algebra or to treat even superficially any of its fundamental concepts and techniques. The only exceptions known to me at this moment are to be found in the recently adopted Danish secondary school mathematics program and the proposals of the Dubrovnik Report. In order to modernize our teaching of algebra, a good deal more is necessary than merely improving our discussion of algebraic symbolism or introducing a little modern descriptive terminology and notation. We need to make an earlier start in treating central algebraic concepts and structures, such as groups and vector systems, and in illustrating some of their important applications to geometry, the solution of linear equations, mechanics, modern physics, statistics, linear programming, and the like. Beyond this, however, we need to present, with adequate illustrations, the conceptual framework and orientation of modern algebra, beginning as early as possible and proceeding by suitable stages. It is precisely here that we will find essential a good philosophical and historical grasp of how algebra has become what it is today.

In attempting to characterize the spirit of modern algebra, I would single out two aspects--the operational and the comparative points of view. At bottom, algebra is exclusively concerned with the performance of operations on elements or objects, the intrinsic nature of which is irrelevant for the mathematical analysis of the operations upon them. The prescription of these elements and the operations in question defines an algebraic system or abstract algebra and fixes its mathematical structure. It is precisely because we have learned to detect and identify such systems in a wide variety of mathematical and scientific contexts that during the twentieth century algebra has grown so much in content and importance. Historically, algebra took its origins in the study of number and quantity, the first algebraic systems to be investigated being the natural number system and the system of positive rational numbers. The mathematicians of ancient Greece obviously were able to identify some of the algebraic aspects of geometry, but just as obviously they failed to perceive that these could be expressed in terms of an algebraic system similar to, but different from, those algebraic systems they knew in arithmetic. Operations upon numbers and operations upon line segments seemed to them to belong to different domains of discourse and they were unable to formulate the complete analogy between the two. Perhaps if the Greek mathematicians had devised suitable symbolisms for dealing with their arithmetical and geometrical concepts they might have been led to abstraction, which stared them in the face over several centuries of active geometrical research. However that may be, we are today in a position to recognize that the introduction of a fairly good algebraic notation during the Renaissance had a remarkably fructifying effect and initiated very substantial advances in the theory of equations and in the understanding of various number systems, as well as in the algebraic treatment of Euclidean geometry. However, it was not until the nineteenth century that the connections between algebra and geometry, first made explicit by Descartes, were truly clarified—especially as a result of the contributions made by Grassmann, Klein, and Hilbert. The geometrical roles of groups, fields, and vector systems were identified respectively in Klein's "Erlangen Programm," in Hilbert's discussion of the axioms of plane geometry, and in Grassmann's calculus of extension. They have been beautifully and ingeniously presented in a novel treatment by Artin, under the title Geometrical Algebra. This short book not only provides the student of mathematics with much food for thought but is rich in materials for anyone engaged in modernizing secondary school or university courses in algebra and geometry.

The operational point of view can probably most easily be approached from the side of group theory and its geometric applications. Starting from examples in elementary arithmetic and number theory, one can abstract the concept of an algebraic system with a single binary operation. Such systems can be understood and constructed at will through their tables of composition, which generalize in an obvious way the various tables the student knows from dealing with the operations of arithmetic. The concept of algebraic systems of this type can then be illustrated by reference to the euclidean motions and their natural mode of composition. The characteristic properties of groups, their verification on the various examples at hand, and the development of their simpler consequences can then be taken up. At this stage it is easy to introduce the allied concepts of ring, field, module, and vector system, and to illustrate them in various ways. It seems to me that it should be possible to teach this much of modern algebra by the end of the secondary school, carrying the discussion up to the point where the polynomial ring over a field can be treated. Of course, a further abstraction is necessary before a general algebraic system can be defined; but this abstraction is almost immediate, involving as it does merely the consideration of an arbitrary number of operations instead of just one or two, and the admission of operations acting on more than just one or two elements.

A considerable part of modern algebra is directed not toward the study of individual algebraic systems but toward the comparison and composition of several such systems. The notions of subsystem, homomorphism, direct sum or product, and quotient system are prominent in such investigations. Since quite elementary illustrations of these concepts are close at hand, some of them can certainly be introduced even at the secondary school level. For instance, there is surely no reason why students at that level should not learn that the most significant property of the logarithm is that it mediates an isomorphism between the multiplicative group of positive real numbers and the additive group of all the real numbers. From a modern point of view, the use of the logarithm for computational purposes is rapidly declining because of the introduction of computing machines, and the significance of its algebraic properties is growing. Since it is never good teaching practice to introduce definitions without showing how they enter into a certain number of important propositions or theorems, some of the elementary results from group theory should certainly be presented in any course where the comparative point of view is discussed. The realization or representation of an abstract group as a group of permutations or transformations, the identification of the homomorphs of an arbitrary group with its quotient-groups, and the connections between abstract rings and the endomorphisms of abelian groups are among the results of this kind which should not be beyond the grasp of reasonably good students of 17 to 18 years of age.

At the present time the comparative point of view is leading to a pronounced interest in a still more sophisticated level of abstraction--the study of algebraic systems in which the object can be interpreted as algebraic systems and the operations as operations upon these systems (such as the construction of a homomorph or of a direct product). Already this new "theory of categories," as it is called, is proving to be very useful in the study of abstract algebraic geometry. Not unnaturally, it is also leading into paradoxical situations similar to those long familiar in the theory of cardinal numbers, which can indeed be treated in some respects as a rather degenerate instance of the theory of categories. I mention this recent development in algebra not because of its immediate interest in connection with the problems of mathematical instruction, but because it illustrates so aptly the way in which algebra grows more abstract, more sophisticated, and more versatile as time passes. Who knows when the theory of categories, already finding a place in research seminars and advanced graduate courses, will have to be taught at earlier levels of instruction? At least, we may observe, there should be no serious difficulty in teaching even relative neophytes that, if algebra feeds on other branches of mathematics to produce interesting algebraic systems, it can also feed upon itself to the same purpose--and with equally interesting consequences!

Let us now go back and reflect a little more deeply on the remarks which have just been made concerning some peculiarly characteristic features of modern algebra. Can we find in this account any central concept which underlies both the operational and the comparative points of view as we have described them here and which may serve in some way to unify them? It is easy to see that we can: the unifying concept is the concept of a function, as understood by present-day mathematicians and logicians. This concept is known by many different names--function, transformation, map--reflecting the historical circumstances under which it emerged independently in as many different branches of mathematics. Yet, after all, it has won recognition as a unitary concept and its omnipresence is a tremendous unifying factor not only in algebra but also in the whole of mathematics. In the case of algebra, as we know, operations, homomorphisms, and isomorphisms are all functions. Now the function concept belongs as a matter of fact to pure logic. There it takes its place as an essentially modern innovation, one of the very few basic ideas in modern logic which cannot be traced all the way back to Aristotle.

In the teaching of mathematics it is evidently necessary to give a prominent position to any such central notion as the concept of a function, illustrating with many examples its widespread occurrence in mathematical and other contexts. It has been customary to introduce this concept at a relatively late stage and to do so then in a piecemeal and uncoordinated fashion, putting off for too long a time its presentation as a single unifying idea. I do not believe that there is any real psychological basis for this hesitant approach, although the early introduction of the function-concept poses complex and obscure psychological problems which we ought to face with courage because of the fundamental importance of the task. It is very likely that we may need to learn more about concept-formation in children and adolescents before we can understand these problems in their true light and find a really satisfactory way of handling them. However, there is certainly no reason why we should not confront children—even very young children—with many arithmetical and geometrical situations in which the function-concept is inherent and thus lead them through early familiarity to an early perception of the common underlying principle. With deeper knowledge of the psychological problems involved, we could almost certainly achieve much better results along these lines, though perhaps none better than could be obtained by an exceptionally gifted teacher. If I were to point to a single problem in the pedagogy of elementary mathematics offering the greatest challenge to our skill as teachers and deserving the most assiduous expert investigation, I would without hesitation indicate the one we have been discussing here. I believe that it should be taken up without delay as a subject of fundamental research.

From a logical point of view, a function is only a special kind of relation and an algebraic system is only a special instance of a more general kind of system which can be defined by prescribing its elements and certain relations, not necessarily operations, involving them. A little reflection quickly shows that every situation studied in mathematics can be formulated in terms of such more general systems. We may therefore refer to them as "mathematical systems" and state the following very interesting and important conclusions: first, that mathematics is just the study of mathematical systems, in the sense described here; second, that all mathematics is expressible in purely logical terms. Thus in a few lines it has been possible for us to reach analytically a position which cost the human race centuries of mathematical experience to discover and to appreciate. The recognition that mathematics is completely abstract and must be accepted as an intellectual activity entirely detached from the physical realities out of which it has grown was forced on us only by the discovery of non-euclidean geometry early in the last century. Until then mathematicians had considered that in geometry and in the calculus they were necessarily dealing directly with concrete aspects of the physical world. However, when it was realized that two distinct and incompatible geometries were available for describing the space in which we live, it was plain that one of them at least must have the status of a purely intellectual creation. We were thus taught the lesson that mathematics transcends experience in a certain sense and were led to entertain the current view that mathematics has no close and necessary ties with the concrete. This is not to say that the applications of mathematics or the interpretation of it in concrete terms does not inspire and guide in a most valuable way the development of mathematics. In fact, our experience shows clearly that if at times mathematics furnishes in advance the framework of some new and fundamental application, there are also many times when concrete thinking about a mathematical problem gives the key to its solution. Nevertheless, the fact is that mathematics can equally well be treated as a game which has to be played with meaningless pieces according to purely formal and essentially arbitrary rules, but which becomes intrinsically interesting because there is such a great fascination in discovering and exploiting the complex patterns of play permitted by the rules. Mathematicians increasingly tend to approach their subject in a spirit which reflects this point of view concerning it.

To understand more fully the connections between mathematics and logic, it is necessary to examine the role of symbolism in the processes of thought. When we introduce special symbols for numbers or for other geometrical or mathematical objects, we easily recognize the arbitrary character of what we are doing. It is not quite so easy to understand that all language has the same arbitrary character and is no less symbolic in this function than the special language we create for our mathematical convenience. It is still less easy to grasp that all thinking is in its very nature also symbolic, because it operates not with the external world but with specific states of the brain which stand for and symbolize experiences of reality. Memory and recall enable the mind to compare, dissect, and combine such symbolic states and thus to effect inference, abstraction, specialization, and the other operations characteristic of abstract thinking. A systematic symbolic treatment of logic and mathematics was not undertaken until the latter part of the nineteenth century. The development since then of various complete special languages strictly adapted to the requirements of logicians and mathematicians has allowed us to base our studies of these subjects upon formal operations or manipulations with strings of symbols. This means that at bottom we have to deal with an algebra in terms of which both logic and mathematics can be expressed; for example, with the algebra of Boole, which also has connections with the theory of sets and, more unexpectedly, with the theory of electrical circuits. In other words, logic itself becomes a branch of mathematics or, more specifically, of algebra. Thus, in a sense which is made precise by the brief discussion, we reach the conclusion that mathematics and logic are coextensive and that both are reducible to an algebra of symbols. This statement emphasizes in the clearest and most cogent manner the abstract nature of mathematics as we have been trying to describe it here.

At the same time this statement allows us to pose further fundamental questions about the nature of logic and mathematics which have important philosophical and pedagogical implications. Already in 1900, David Hilbert had included, in his celebrated list of important unsolved problems in mathematics, the problem of finding a theory of mathematical proof which would establish the consistency and the completeness of mathematics; that is to say, the possibility of deciding without contradiction the truth or falsity of every properly formulated mathematical proposition. Some twenty years later he began to discuss this problem in a systematic way with special reference to the foundations of arithmetic. It was then not long until Gödel showed that the logical problems of arithmetic, as made precise by Hilbert and his school, have only negative solutions, consistency implying incompleteness—that is to say, the existence in arithmetic of undecidable propositions or, more precisely, propositions which cannot be proved either true or false in terms of the assumed underlying logical apparatus. Moreover, Gödel showed that this phenomenon is of very general consequence and can hardly be avoided in any obvious way. He thus laid to rest Hilbert's dream. Almost at once Church and Turing discovered a somewhat similar complication in the theory of proof by showing that even if there exists a proof of some arithmetical proposition, it may not fit into any prescribed scheme of proof. In short, no routine or mechanical procedure for constructing proofs in arithmetic will cover every case where it is possible to produce a proof. Again the results of Church and Turing are of very general bearing, having applications far beyond the boundaries of arithmetic. Thus two beliefs which had been held implicitly by all, or nearly all, mathematicians from the earliest Greek times to the present were placed in a new and troublous situation. It is no longer possible today to have faith that one can find, for a given mathematical discipline, a complete logical organization of its propositions into the true and the false or that one can produce, in an automatic or mechanical way, formal proofs of those propositions which are provable. This does not mean that for a restricted kind of proposition in a particular discipline a complete logical organization is impossible or that the systematic construction of proofs for the true propositions among those admitted to consideration is beyond our reach. On the contrary, Tarski has shown that for a wide category of propositions of Euclidean geometry, as based on a particular set of axioms, there is a decision procedure.

This category embraces the propositions which one would normally consider as the most natural and interesting in plane geometry. Nevertheless, it must be expected that in the axiomatic treatment of mathematical systems the categorical set of axioms will be rare. The general situation will be one in which it is possible to augment the axiom scheme by adding without limit new axioms which lead neither to inconsistency nor to categoricity. On the other hand, the results of Church and Turing leave the mathematician a much wider scope than he would have if there existed some mechanical device for constructing proofs of all true propositions in a mathematical discipline. The fact that the construction of proofs must thus remain an art rather than a routine technique shows how important a role is always to be played by the gifted creative mathematician.

From a pedagogical point of view there is an antithesis between the manipulative aspects of mathematics—that is to say, the correct, and at bottom, mechanical calculation with mathematical symbols—and the intuitive search for the patterns or structural features latent in particular mathematical systems. In the past great emphasis has been laid in the teaching of mathematics upon the mastery of manipulative skills and ingenious manipulative tricks. Drill, far too often of the dullest kind, has accordingly come to occupy a far too prominent place in mathematical programs. At the present time there is a strong tendency to increase the attention paid to the structural side so that students may better understand the basis for the manipulative procedures they are required to learn and at the same time may gain some insight into the structure of the principal mathematical systems encountered in algebra, geometry, and other branches of mathematics. The teaching of manipulation as such runs the risk of extinguishing the mathematical interests and imagination of the better students in dull routine and sheer drudgery. There is considerable evidence that the acquisition of manipulative skills is promoted far better by the cultivation of structural insights than by an exclusive appeal to systematic drill, necessary though the latter certainly is. In short, a modern program in mathematics must resolve the antithesis between manipulation and intuitive insights by giving appropriate places to both and making each support the other. We cannot remain content with training our students to calculate speedily and accurately or to solve an interminable sequence of tricky geometrical problems, though for obvious reasons we must not neglect this side of their mathematical training. Success in both pure and applied mathematics depends upon the ability to calculate and to manipulate the symbols of mathematics. We must also strive to develop an equal skill in the search for patterns and structure, beginning at the earliest possible stages in the school curriculum.

In recent times the development of remarkably speedy and versatile electronic calculating machines has completely altered the art of computation. In consequence, it has also modified the sort of training in computation which should be given in schools and universities. In principle, it is now possible to carry out every kind of mathematical or logical manipulation by machine processes far more speedily and far more efficiently than anything even the most skillful mathematician can do. Elaborate computations which were previously quite out of the question have now become altogether practicable. Consequently, in many fields where mathematics finds application, there is a heavy and fast growing demand for high-speed electronic computers and for mathematical specialists capable of using them with skill and efficiency. When we teach the art of computation to our students, we should therefore lay much less stress on training them to function as calculating machines and much more on preparing them to become knowledgeable and intelligent users of the various mechanical and electronic computing aids now available, beginning with the slide rule and the manually operated desk computers. Of course the mathematicians who intend to direct or supervise the use of the high-speed electronic computers require a very extensive knowledge of higher mathematics and logic and must receive their technical training in the university.

In this discussion of current developments in mathematics, special stress has so far been laid upon the algebraic and logical aspects. In order to give a more complete description of the present state of mathematics, it is necessary to indicate something of what is known about mathematical systems which are not purely algebraic systems. We may observe at once that we have not yet gone very far in exploring the possibilities inherent in the extremely broad definition of a mathematical system which has been given here. The fact is that the only kinds of non-algebraic structures which have been investigated in a systematic way are those characterized by the presence of an order relation. The study of ordered sets and their role in mathematics is essentially a recent development, even though the concept of order has been part of man's intellectual equipment from prehistoric times and has been recognized from the beginning as basic in both geometry and arithmetic. Topology, with its central notions of limit and continuity, may be defined most simply in terms of order and cannot be understood without using the properties of ordered sets. The discovery and development of topology during the present century illustrates in a most striking fashion how rich and intricate the implications of an apparently very simple and primitive concept such as that of order can be. What marvels may await us when we undertake to explore other apparently simple and primitive relations and the mathematical systems involving them? We can only guess that they will make mathematics as we know it today look like child's play!

When the possibilities of algebraic and ordinal structure are combined, as is done in the study of topological groups and topological algebras, the foundation is laid for an extensive development, the beginnings of which are a characteristic achievement of twentieth century mathematics. Indeed, a detailed account of what has been accomplished in the last fifty years in this field alone would give a very good idea of the nature of modern mathematics and the technical progress which can be claimed for it. One of the most significant advances of twentieth century mathematics, for example, is the recognition of the role played in mathematical analysis by the abstract concepts of topological algebra and the development of functional analysis as a powerful tool for studying the problems of classical analysis. It would be quite accurate to say that analysis has become a part of topological algebra. In fact, until we look to other possibilities lying beyond algebra and the theory of ordered sets, it seems quite clear that pure mathematics will be the study of topological algebra, taken in the broadest sense of the term.

From these very general considerations of the nature of mathematics, as we understand it today a little more than halfway through the twentieth century, there emerges a strikingly clear view of the essential unity of mathematics. If it were possible in this discussion to examine in detail the technical progress in the various branches of mathematics since 1900 and the way in which they have been more and more closely bound together by the results of modern research, we could do no more than illustrate and confirm what can be made apparent on a priori grounds. In teaching mathematics, even at the most elementary levels, one of our chief goals must be to lead our students to an early awareness of this underlying unity. The greatest disservice we could do them would be to maintain the compartmentalization of mathematics in unrelated disciplines in order to concentrate our efforts upon the inculcation of isolated manipulative and technical skills.

Having reviewed at some length the characteristic tendencies of modern mathematics—its abstractness, its unity, its fusion with logic, and its current aspect as the study of topological algebra—and having alluded, albeit in insufficient detail, to its technical advances and to the extraordinary proliferation of its applications during the twentieth century, I should like to close with a few words about the implications of this overall view for the teaching of mathematics. In doing so, I wish to emphasize especially that it has become necessary to teach mathematics in a new spirit, consonant with the spirit which inspires and infuses the work of the modern mathematician, whether he be concerned with mathematics in and for itself, or with mathematics as an instrument for understanding the world in which we live. To be sure, we must enrich and reorganize the mathematics curriculum, making accessible to our students, as early and as easily as possible, our greatly increased store of mathematical knowledge, both pure and applied. This requires a careful analysis of the curriculum in the light of current needs and of our better understanding of the different branches of mathematics and their interconnections. Thus, for example, we have pointed out how important it is that we should begin teaching the concepts of modern algebra at a much earlier stage than at present. On the applied side, we should point out that it is equally important to introduce at an equally early stage the fundamental concepts of probability and statistics, which have to be recognized as the cornerstone of applied mathematics, even of modern physics which has had to abandon the deterministic views of its earlier days. A rather obvious comment to make upon the contemporary discussions of the mathematical curriculum is the considerable emphasis that they place upon a more explicit and detailed development of the logical aspects of mathematics as part of elementary and secondary instruction. This emphasis is quite necessary. It is important for young students of mathematics to understand the need for precision and the nature of proof. Eventually all mathematicians and all teachers of mathematics need a clear understanding of the nature of logic and of its relation to mathematics, but this must come, in my opinion, at a stage well beyond the secondary school. Finally, it is important for even the neophyte in mathematics to grasp the axiomatic method. This is just as true for the future applied mathematicians as it is for the pure. In fact, the construction of mathematical models for various fragments of the real world, which is the most essential business of the applied mathematician is nothing but an exercise in axiomatics. A very cogent reason for insisting upon an axiomatic treatment of geometry at the secondary school level is that geometry is the only subject in elementary mathematics which has a palpable connection with the real world and provides a good example of model construction. The same cannot be claimed for arithmetic or algebra, as a moment's reflection will show. In the majority of discussions about the axiomatic treatment of geometry, this aspect of the matter is not sufficiently emphasized, probably because mathematicians are primarily interested in the logical side of the question, as the frequent insistence upon the use of the word "postulate" instead of "axiom" clearly indicates.

When an acceptable modern curriculum has been shaped in terms of its mathematical content, one must still be concerned with the spirit which animates both the subject matter and the manner in which it is taught. It is here that it is highly appropriate to demand that, even in the earliest stages, an effort should be made to bring out both the unity and the abstractness of mathematics. To be sure, neither unification nor abstraction, as the very terms indicate to us, can spring from anything but specific experience—but from experience which may be either physical or intellectual. Hence in building up to the unified and the abstract, we must begin by offering our young students just such experiences. Indeed, it is a great challenge to us as teachers to find new types of experience, even on the physical level, from which our youngest students can start their approach to such abstract ideas as that of a function as a transformation. In addition to being concerned with these characteristic aspects of modern mathematics, we must be equally engaged by the effort to present mathematics as a creative, artistic activity of the human spirit. No matter how extraordinary may be the performances of our mathematical machines, these are but our creatures—and, as the analyses of Gödel, Church, and Turing suggest, creatures they are likely to remain. Accordingly, an important part—perhaps even the most important and the highest part—of our task as teachers is to foster in our students, from their very first day in the classroom, that creative spirit without which mathematics would become sterile and die.

One final word. It is a deep fact of human psychology that the thinking of children tends to be more abstract, more imaginative, and more creative in many respects than that of adolescents. We should grasp the wonderful opportunity this gives us for initiating children at a very tender age into the true spirit of modern mathematics and leading them through the elements of arithmetic, algebra, and physical geometry before they enter secondary school. While we are primarily concerned in this conference with secondary and university mathematics, let us not forget that whatever we can do at these levels has to be based upon what we have already done in the elementary school to set our pupils' feet upon the road to mathematics.

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Discussion Following Addresses by Professors Choquet and Stone

The addresses by Professors Choquet (France) and Stone (U.S.A.) gave rise to some profound questions involving the psychology of learning mathematics.

In response to an inquiry by Professor Laguardia (Uruguay), Professor Stone indicated that he believed that the development of creativity and imagination in the child was as important as the logico-deductive aspects of the teaching of mathematics. Professor Laguardia then asked Professor Stone for concrete suggestions for cultivating this creativity. The exchange of ideas on this topic is given below:

Prof. Stone (U.S.A.): I have never taught small children, not even students at the secondary level. My only experience comes from the time when I was a child. I believe it is a matter of patience of trying one way and another, always analyzing the results.

Prof. Laguardia (Uruguay): In this connection, I would like to describe a procedure which I have tried successfully with students in the last year of the secondary school. The exercise consists in giving the student a problem and requiring that he change some conditions in the hypothesis and that he make all necessary changes in the corresponding conclusions, justifying each alteration as he proceeds. I believe this to be an effective method for developing the critical and analytical abilities of the student.

Prof. Coleman (Canada): I would like to ask Professor Begle to comment on experiments in schools of the United States with children of the elementary schools.

Prof. Begle (U.S.A.): It is a matter of trying to show the very young children the rewards of the intellectual effort which is necessary to describe nature in mathematical terms.

Prof. Santaló (Mexico): I believe that one of the important themes in Professor Stone's address is that emphasis was given to the purely manipulative side of mathematics in the old methods of teaching. It is my opinion that this error not only is to be found in the past, but is also one of the greatest deficiencies in the method of teaching in vogue at the present time. I believe, moreover, that it is one of the principal causes of the bad quality of teaching observed everywhere. School examinations, as a general rule, are directed to the evaluation of this manipulative dexterity of the student, which seems rather absurd to me.

The discussion ended with this remark by Professor Santalo, with which the participants seemed to be in wholehearted agreement

*SOME OBSERVATIONS ON THE TEACHING OF MATHEMATICS IN THE UNIVERSITY**

Professor Guillermo Torres (Mexico)

It has often been said that the teaching of mathematics is confronted with grave problems, an examination of which always points to the same source. It is said, for example, that because of custom and tradition, the content of mathematics courses is petrified, and that many years will elapse before the university student will encounter an idea that has not been known for centuries. Nevertheless, we speak quite properly about the constant and explosive development of mathematics and its ever-increasing influence on any field related to reasoning. Thus we acknowledge, on the one hand, that the teaching of mathematics is inadequate but, on the other hand, that mathematics has more vitality today than it ever had. Can it be that there are capricious aspects of education that have not yet been recognized?

The above statements are made as a confession of faith. A confession of faith in the future of mathematics, in its irrepressible vitality, in its power to transform scientific thinking in heretofore unsuspected ways. Poor teaching of mathematics must not be permitted to keep mathematics from making its influence felt in an ever-widening sphere of our culture.

Nevertheless, it is true that methods of teaching and curricula have not changed at a fast enough pace. The gap that separates concepts and attitudes acquired through the usual program of study and the ideas and attitudes that a modern mathematician needs for his work is becoming wider each day, and it is therefore obvious that teaching procedures must be modified.

I shall not refer to difficulties of a practical nature that will be encountered in attempting to modify the method and content of mathematics courses, both of which are strongly rooted in tradition and custom. I prefer to consider the problem of deciding what should be taught and the best way of teaching it.

It is evident that not everything that is taught is useless, but it is also certain that the student of mathematics spends many hours of his life acquiring knowledge that in no way contributes to his

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*Address translated from the Spanish.

scientific background. It will be necessary, therefore, to decide what material should be omitted and what should replace it. In the course of discussions on this very matter, my colleagues have, on occasion, expressed very radical opinions, and sometimes I have observed a tendency on their part to consign to oblivion whole sections of classical mathematics which constitute, without doubt, masterpieces of scientific thought.

An example will serve to illustrate this extreme position. Suppose that a young man who has just entered the university and who has studied ordinary subjects such as algebra and elementary geometry, analytic geometry and calculus, and nothing more, is to be taught the meaning of "a space T of vectors tangent to a differentiable variety of class $\dot{C}$." We begin by describing a topological space and then the task gets easier. It is a matter of a set and a family of open subsets which satisfy..., etc., etc. Then we can tell him what an n-dimensional vector space E over the real numbers is. Then we define "norm" and along with it the topology of R. (This can be done because we have previously defined "topological space.") Afterwards we can give the definition of the differentiable homomorphism of class C^* between two open subsets of E. Now we are ready to define a differentiable variety, a vector tangent at a point, and to draw a conclusion as to the topology of T. All this can be done in an hour, and if the student is patient enough, he would most likely have understood the formal definitions. Have we taught him anything? Evidently the answer is "No."

I intentionally exaggerated in the above illustration because it is clear that for a student to accept a differentiable variety as something quite natural, it is necessary for him to be familiar, for example, with certain concepts of classical differential geometry concerning superficies imbedded in E^2. Many examples like this one could be given.

I am convinced that the new ideas that a student acquires should be acceptable to him as being quite natural. Therefore I cannot see any other way of teaching mathematics except by following, more or less, the path which mathematics took in its historical development, omitting anything that is considered to be of little importance. The presentation of mathematics in its exclusively formal aspects would give it the appearance of an inhuman activity that had no reason for existence.

Nevertheless, it seems that this formal and "dry" method of presenting mathematics as a succession of definitions and theorems is being recommended more frequently. For some reason, the mathematician hesitates to write about his vital experiences as a creator. He confines himself to presenting the results of his investigations in a polished form and rarely does he divulge any hint as to his method of working, his hobbies, his interests in science, and even less about his intellectual training. The mathematician writes mathematics, but he rarely writes about mathematics.

As a practical method for deciding what must be included in mathematics education in the university, permit me to suggest the following for your consideration. Ask one of the most distinguished mathematicians what, in his opinion, the young man must know in order to arrive, by the shortest possible path, at an adequate mastery of the field for which the mathematician was distinguished. Rest assured that if we do this, we shall meet on common ground and this will permit us to decide what must be considered as fundamental in the training of the mathematician. As for methods which we should follow in teaching, I have not discovered one idea or suggestion which might be of interest.

NEW IDEAS IN THE TEACHING OF MATHEMATICS IN THE COLLEGES OF THE UNITED STATES

Professor E. J. McShane (U.S.A.)

In the addresses yesterday we heard an extremely interesting description of the development of mathematics in this century. It is a beautiful and powerful development, and its bounds are not in sight. But no matter how superhuman mathematics may seem, in one essential respect it is tied to us humans: if we do not pass it on to future generations, it will die and be buried in the yellowed pages of unread tomes. And the greater the growth of the body of known mathematics the more difficult the problem of doing the best possible for the students. Mathematics itself knows nothing of national frontiers. But each one of us professors teaches in a certain university in a certain country, and we have to conform to the boundary conditions imposed by local circumstances. In this address I shall say something about present conditions in the United States. I shall outline some changes that the mathematicians of the United States (or at least some of them) hope to see in a few years, and I shall indicate the little that we have as yet been able to do.

In the United States, the typical high school graduate is about 18 years of age, and in twelve of those years he has attended classes. If he wishes to continue, he must make a choice. He can enter a college of education, that is, of pedagogy; or an institute of technology or college of engineering; or a college of arts and sciences. It is these last that we call simply "colleges," for the sake of brevity. It is necessary to change the subject for a moment in order to explain the word "college." I have been using "colegio" as a translation of "college," but the "college" of the United States is entirely different from the "colegio" of other countries. It could be described in this way: Suppose that in a Latin-American university some of the faculties were separated from the others and joined together in an administrative unit; among these would be the faculties of arts, languages, and literature, mathematics, physics, chemistry, and biology, but not those of law, medicine, engineering, or pedagogy. Suppose, furthermore, that the first-year student were to enrol, not in any one faculty, but in the union of faculties thus formed, and that at first he were to study subjects in several faculties, and only in the last years would he choose most of his courses in a single faculty. The result would be the "college" of the United

States. The college of engineering is another such union of faculties, which we do not need to specify; the college of education consists of the faculty of pedagogy. When I speak of "colleges," you will understand that I mean "colleges of the type found in the United States."

One can see that by the very nature of the college it will contain students of the most diverse interests. Four years later, some of these students will finish their studies and enter business or politics or some other occupation. Others, after the proper training, will be lawyers or doctors. Others will be the scholars, the philosophers, the scientists, the mathematicians, and the learned men in all fields. And the colleges themselves are of very different types. Some are independent and exist as separate bodies (and the same can be said of some technological institutes and some colleges of education), others are parts of small or large universities, which also contain faculties of graduate studies, of medicine, and so on.

In some outstanding universities, it is practically impossible to gain admittance without having taken all the courses in mathematics in the secondary school and having made excellent grades in them. In other universities, in particular in some state universities and in many colleges, all that is needed is a certificate of graduation from any secondary school. Often the entrance requirements include two years or two years and a half of high school mathematics. Thus it is plain that there will be students of very assorted capacities and degrees of preparation.

Although the colleges of the United States differ so much from each other, in one essential feature they are almost all alike; in their first two years the students are expected and, in fact, required to attend classes in an assortment of subjects. In their third and fourth years, the students are separated in accordance with the choice of field of specialization. But the beginners are all together, and even if they wished to do so, the professors could not distinguish between the future biologists and the future mathematicians. In many universities, the beginning students of engineering are together with those of the college, and likewise those in education. In any case, it is impossible (and I think it would be a mistake) to teach modern pure mathematics to the future mathematicians and leave slightly old and dirty mathematics for the others. If we really believe that the modern developments are important improvements, they ought to be advantageous for the applications, too. And I believe firmly that this is true; but there is always a danger that our enthusiasm for new ideas will lead us to teach them in place of others that may appear banal to the mathematicians, but seem essential to our colleagues in other specialties.

Perhaps what I have been saying to you may appear to be a description of chaos. Permit me to say a few words in favor of this chaos. The situation is the natural result of the axioms of education in our country. We wish to offer to everyone the opportunity of being educated up to his own limits, and we wish to offer the greatest possible freedom in the choice of a career. In view of the millions of pupils and students, it is clear that we have created a problem that is not easy to solve; but we are firmly convinced that it is better to live with such a problem than to avoid it by abandoning our axioms

Now let us look at the college mathematics of thirty years ago. We usually started with a semester of "college algebra," meaning a collection of assorted topics chosen for usefulness, but disconnected and disordered and with no trace of the logical structure that gives beauty to mathematics, new and old; in the second semester, trigonometry (including a multitude of computations) and analytic geometry; then a semester of differential calculus and a semester of integral calculus, carefully distinguished. The book most often used was that of Granville. I do not wish to ridicule that book. In comparison with those of the first years of the century, it is excellent, with a coherent organization and a fine collection of problems. But by today's standards, its proofs lacked rigor, and of course it contained nothing of the modern idea of what mathematics is. I cannot assert it with certainty, but it seems to me that the algebra was taught to give the student manipulative skill that he should have brought with him from high school; the trigonometry because of the real or imagined insistence of the engineers; and the analytic geometry because it was supposed to be essential for calculus. With calculus began the genuine mathematics, which would serve as the foundation for the structure of advanced mathematics; nevertheless, the attention was usually fixed on solutions of routine problems and not on the general ideas that could make it possible to solve new kinds of problems. The best mathematicians knew that the situation was bad, but considered it beneath the dignity of a leading mathematician to write a college textbook, and even more to give any thought to problems of high school teaching.

I do not know just when attempts at improvement began. Over twenty-eight years ago, instruction at Princeton began with calculus; they used a book of dubious quality because that is all there was. (In 1934 appeared the English translation of Courant's Differential and Integral Calculus, which in my opinion continues even today to be one of the best books on the subject; but it was seldom used for beginning calculus, and so its publication did not help matters much. Between 1942 and 1948, there were definite signs of awakening. The first change that was attempted in several universities met with little success, and we need to look at it with some care. I do not wish to discuss whether mathematics is part of logic, or vice versa, or neither the one nor the other. It is indisputable that mathematics cannot exist without exact logical reasoning. Nevertheless, in college textbooks there was no discussion of logic, and too often there was not much use of it. The obvious remedy was to introduce a few weeks of study of logic at the beginning of the first year. As a result, several books appeared that began with at least a chapter on logic. The students had to check the validity of statements such as "If every oojum is a boojum and no boojum is a woojum, then no oojum is a woojum." We hoped that by means of the study of pure reasoning, by means of meaningless expressions, the students would be able to get used to thinking in a logical way. I myself adopted this method, at first with enthusiasm and later with disenchantment. The students saw nothing but foolishness in such words. We professors knew that logical thinking is indispensable for mathematics and the sciences, and beyond the exercises we could see the uses for which they were intended. But the students naturally could not anticipate uses which they would not meet until later, and they rebelled. Moreover, our colleagues in engineering and the sciences could see nothing in these exercises but a waste of valuable time. And in the long run, they have won. There are few universities in which mathematics now begins with a study of logic, pure and devoid of immediate application. Instead, the authors of the best books for beginners in mathematics try to introduce logic by its use. The present opinion of the majority of mathematicians is that if we show the student many examples of rigorous proofs, and if we avoid with utmost care all false or unclear proofs, he will learn to think in an acceptable way, that is, logically. Formal logic, which fundamentally is the analysis of thought-patterns that we consider correct, can be postponed until later.

I mention this failure, not to ridicule it, but because we should learn from our mistakes. When we try to introduce something new in teaching and encounter resistance, we should neither give up the attempt nor insist on keeping every detail we have thought out. It is always possible that we might attain the essential part of what we want by other means that would not meet the same resistance.

In 1953, the Mathematical Association of America formed a committee, the Committee on the Undergraduate Program (or C.U.P., for the sake of brevity), for the purpose of undertaking a reorganization of the teaching of mathematics in the four years of college. If I mention this committee frequently, it is not because it had great influence or much success, but because it was a collection of mathematicians who were seriously interested in the improvement of mathematical education and because it served as a focal point for collecting the opinions of other interested persons. Perhaps I exaggerate, but it seems to me that the opinions expressed in the committee itself, and in the several conferences it organized, serve to characterize sufficiently well the ideas of the majority of the mathematicians of the United States. I must admit that among my ideas on the teaching of mathematics, I can no longer remember which are the few that I might call my own and which are the many that I have borrowed from the committee. At least, I think I speak for many other mathematicians, although in no respect do I speak for all of them; we mathematicians are decidedly individualistic, and no one can dare to speak in the name of all of us.

Now let us look at some of the agreements reached by the committee. In the first place, it is the consensus that everything called mathematics should be mathematics in the present sense of the word; that is, it ought to consist of a mathematical system or systems, beginning with logic and proceeding logically to conclusions. Even here we are not on perfectly solid ground. There is no unanimity on what constitutes a logical proof. Perhaps I should say no more than this: the proofs should seem logical to the majority of mathematicians in the present year. If we should require unanimous agreement, we would be far more demanding than the editors of mathematical periodicals. If we say that a proof will appear rigorous in the year 2061, we are setting ourselves up as prophets; it is possible that our ideas of rigor may appear laughable in that year. We must not boast of infallibility.

Having said this, we have to retreat a little. There are many details in mathematics that seem evident to a beginner, but in fact follow from the axioms only by way of a rather long proof that may be too subtle for a first-year student. For example, if a young man has just met the idea of a continuous function, he will regard it as evident that a function continuous on an interval cannot assume two values without assuming all values between those two. If we stop to prove this, the ordinary student will merely think that we are splitting hairs. We do him no harm if we simply say that this proposition, which seems evident from the figure, can be rigorously proved, but that we are postponing the proof until a more advanced course. But each time we say this, we must be sure that we are not lying. We must never leave gaps that we cannot fill in if asked. Above all, we must never expose the student to false assertions. It may seem that it is not worth the trouble of saying this, but there are too many books in which such errors are found. An interesting example is a well-known book of modern physics which on page 30 presents us with what is called a proof of a theorem, and on page 31 explains why it is false.

In the second place, we must not merely place ideas before the student; we must communicate these ideas to him. It is possible to construct a chain of mathematical propositions that a student can understand, but if he cannot see any relationships between them and things he already knows, he will not accept them. He will regard them as a diet of straw, no more useful than the game of chess and far less interesting. It does not help much to promise him that in the distant future he will be able to use part of what he is studying, especially if he knows that we look down on applications. It is useless to tell him that mathematics is interesting when all that he sees of it is dull. And in truth, if we present him a subject as interesting and useful as mathematics in such a way that he cannot see anything either of its beauty or of its utility, the fault is ours.

Each new idea should be presented as the abstraction of ideas already encountered in concrete situations. It should deserve the respect of the student because of its elegance, and it should lead to applications of undoubted importance.

In regard to applications, I ask your permission to digress a moment. I believe that contact with applications is of outstanding importance, not only in teaching, but in all levels of mathematics, up to the highest. Mathematics is neither the slave of the sciences nor their master, but their perpetual ally. When mathematics is nothing more than the servant of the sciences, and is taught only in order to solve problems that originate in the sciences, in the long run it suffers the fate of all ill-treated slaves; it loses its beauty, seems to lack intelligence, and is unable to attract the interest of any reputable man. On the other hand, when mathematicians insist inflexibly on their independence and disdain those who have an interest in applications, they close off an important source of new ideas. Permit me to read some words of John von Neumann (which I abbreviate slightly): "I think that it is a relatively good approximation to truth...that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one....But there is a grave danger that the subject will develop along the lines of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration....In any event, whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more of less directly empirical ideas...."

To show that the separation between mathematics and the sciences may not be an illusion, I quote some words of Professor Borsellino: "I think it is one of the gravest dangers in the relationship between physicists and mathematicians at the present time that we do not understand each other on the research plane....Essentially mathematicians are now working on subjects that we as physicists know very little about. We should make contact with them at some points to permit real communication from modern mathematics to the physical knowledge of the average physicist...."

I am firmly convinced that if mathematics abandons all contact with applications, it cannot avoid the fate that has already befallen the classical languages. In the Middle Ages, Latin was the means of communication among educated men in all of Europe. But in the Renaissance, it was recognized that this language was no longer that of the best Roman authors. The scholars eliminated the errors and brought the language back to its ancient perfection. Since then, the Latin language is free of the medieval solecisms--and is dead. Mathematics is like the giant Antaeus. It will die if some Hercules can lift it into the air to avoid all debasing contact with the ground. Antaeus did not need to fall down and remain stretched out flat. I believe that mathematics too should have its feet on the ground and its head in the clouds.

Let us come back from the clouds to the beginners. For most of them, mathematics, no matter how correct it may be, is merely dry straw if it cannot be related to something of more evident interest. The things we learn are remembered by being interconnected with other things that we already know. New ideas will more readily attract interest and will be more easily remembered if they are introduced by means of applications to familiar situations and are soon applied to other problems.

As to applications, we should keep in mind that mathematics is beginning to enter into sciences that previously got along without it. The traditional applications are in physics and engineering. We ought to look for interesting applications in the new areas.

If I say that we must teach abstract mathematics, I say nothing. By its very definition, everything that can properly be called mathematics has to be abstract. But obviously there are different degrees of abstraction. To devotees of abstraction, it seems elegant and economical to begin with the most general and abstract form of the mathematical discipline under discussion; from this all the other forms can be deduced, being merely special cases. But the human race did not arrive in this way at our present-day abstractions. Many mathematicians found great pleasure in the discovery of these abstractions. The student is already too willing to believe that mathematics has come down to us from centuries past and that there is nothing more to do than to learn the theorems in the books. We do him harm if we tell him, "Here is the perfect form, completely general and completely abstract. Take it, and learn it." Instead, we should teach him how to generalize and how to abstract.

These are the conditions that should be met by the program of mathematical teaching in our colleges (or at least, this is my belief). And now we ask ourselves, is it possible to design a program which at least comes close to satisfying these conditions?

Let us again remember the astonishing diversity of the first-year students and of their interests. Let us remember, too, that the majority of them will take only one year of mathematics, while others are the mathematicians of the future. The problem that we wish to solve is this: To offer them a first course of mathematics that will satisfy the conditions of which I have spoken, that will be new and interesting for the better students, that will not require more knowledge of mathematics than most of the students have, that can be applied easily enough to be useful to those who are going to specialize in subjects calling for only a small amount of mathematics, that can serve as foundation for more advanced mathematical courses, and that is worthy of being part of the education of every cultured man. I do not mean to say that there are no other problems of mathematical education, but for the time being, this will be amply sufficient.

The members of the C.U.P., being mathematicians, began by proving an existence theorem. We did not consider that it would be enough to describe a course that in our opinion would be satisfactory. We had to construct such a course. And in the years between 1953 and 1958, the C.U.P. published four books that constitute at least a partial solution of the problem that I have just mentioned, and not just for the first year, but for the second year also.

There is general agreement that the calculus has great importance. If we decide to begin the first year of college mathematics with calculus, it would be easy to satisfy all the conditions but one. All we would need to do would be to use Courant's book. But no matter how outstanding it may be, this book is beyond the reach of first-year students with no more than average preparation. The C.U.P. set itself the task of writing a calculus text that would comply with the conditions previously mentioned, that would not assume any previous knowledge of trigonometry or of analytic geometry, and that nevertheless would contain enough to satisfy the needs of, say, biology and economics. The result, being intended for all first-year students, was called Universal Mathematics, Part I. It was meant to be, not a textbook, but a source from which any mathematician who wanted to write a textbook could take as many ideas as he wished. On the left-hand pages appears what the students are supposed to read. Occasionally the authors take refuge in "From the figure it can be seen that..." or in "The proof, although possible, is too difficult for this book, and we postpone it until a more advanced course." But there is none of this on the right-hand pages. There the proofs are complete, except perhaps for errors that may be present because the authors are fallible humans. Thus the students who do not wish to accept statements on the basis of authority (and may there be many of them!) can convince themselves that the proof is indeed possible. And not only the earnest students, but the members of the C.U.P. itself, know that there is no error or omission, modulo human fallibility.

This book was used in several classes, with partial success. Some parts were too difficult for practically all the students, for example, the Moore-Smith theory of limits. The professors who made the test announced the results in the American Mathematical Monthly. But I think that today there is no college in which the book is being used.

The second part of Universal Mathematics differs greatly from the first in subject matter. It is written for future economists, and others, who previously received little attention in the teaching of mathematics. Here are some of the topics treated: The algebra of sets, Boolean algebra, implication and quantification, induction, finite sets and enumeration of them, measures (just a little of this!), probability (limited to discrete probabilities and including the binomial distribution and conditional probability), something of the theory of groups and of fields, up to the introduction of complex numbers and the solution of linear and quadratic equations.

Those who are familiar with Finite Mathematics, by Kemeny, Snell, and Thompson will recognize that it has much in common with the book I have just mentioned. This is hardly surprising. Kemeny was a member of the C.U.P. and exchanged ideas freely with the other members. Since the book by Kemeny, Snell, and Thompson is well known, we can consider that many of the ideas of Universal Mathematics, Part II are available to every instructor.

In order to begin with very little and reach the calculus in one semester, it was necessary in Universal Mathematics, Part I to limit ourselves to functions on the real numbers or on intervals of real numbers. The study of functions of several variables is left for the second year. For this year, the C.U.P. proposed a separation of the students. Those interested in mathematics or physics or engineering would take a course that would be a logical continuation of the first-year course, but not radically different from those now in existence. For those planning to specialize in the less quantitative sciences, these traditional courses are not very appropriate. There is no book written primarily or even secondarily for such students. Because of this, the members of the C.U.P. felt that they had the duty of writing textbooks to serve that purpose. In 1958 two books were published, for the two semesters of the second year. The title of the pair is Modern Mathematical Methods and Models; the first volume is called Multicomponent Methods and the second

Mathematical Models. In the first volume there are three principal topics. There is a study of matrices, operations with them, and their applications to systems of linear equations and to linear transformations in spaces of finitely many dimensions. There is an explanation of the use of tables for integrating complicated expressions and solving differential equations. Finally, by way of the theory of extrema of functions of several variables, we reach linear programming. The second volume contains more of the theory of probability, up to Markov chains, and concludes with von Neumann and Morgenstern's theory of games.

We still do not know if this program is too ambitious. To the best of my knowledge and belief, no one has used these books as textbooks. In the social sciences there is a growing recognition of the importance of mathematics, but it has not yet grown very much. Perhaps some time in the future, a book more or less like that of the C.U.P. will be used, but we have to wait.

And now let us move on from what should be and what could be, to what is. First I will say something about the changes in the teaching of the traditional courses and then something about the changes in the curriculum. From what we hear at meetings of the mathematical societies, we know that practically everywhere there is at least the intention of improving teaching. Obviously we know little about what goes on in other people's classes. But we can draw some conclusions from what we see in the textbooks they are using.

The recent calculus texts are mostly much more rigorous than those of twenty years ago. Some written for the first course in calculus and some for the second course are in fact introductions to the theory of functions of real variables. Most of these demand more mathematical maturity than typical first-year students possess. Thus we must classify them as books for students of mathematics and of the sciences in which mathematics is recognized as highly important. It is not here that we can hope to find the newest ideas. But there are many other books that have been written for the horde of first-year students. Even with these we have not entered the promised land. Often I have asked friends, "What calculus text are you using now in your university?" and they tell me. Then, "Do you like it?" and they say, "Not entirely." It is natural that those who want to introduce new ideas should choose notations very different from the old ones, so as to avoid confusion with the errors that they are trying to eliminate. Thus, some of the new books look astonishing to conservative mathematicians, and may perhaps appear a trifle exaggerated even to those who are not conservative. The apostle of the new ideas can easily make the same sort of mistake as a colleague of mine who wrote the manuscript of a book on exterior ballistics. His plan was never to use any one symbol to stand for two different concepts. To reach this goal he had to use so many different symbols that he ended by using the Eskimo alphabet. It was impossible to misunderstand the symbols. Besides this, it was impossible to read the book. Such purism is an exaggeration. In mathematical periodicals we frequently make use of expressions that are not rigorously justifiable, because it is convenient to do so and we know that the readers will understand what we mean. For example, the space L_2 consists of certain equivalence classes of functions. Nevertheless, we often say that some function or other belongs to the space L_2, which is impossible; by this we mean to say that the class consisting of all functions equivalent to the given one is a member of L_2. What we say is wrong, but it bothers no one; the reader can easily furnish the small correction that is needed, and he does not have to read an involved sentence. Sooner or later there will be books that will present the calculus to the student in a way that is both correct and easily understood, without seeming baroque to anyone.

With regard to the new courses, there is a great difference between the small colleges and the large universities. The C.U.P. has just finished an investigation of the accessibility of new courses to students in the colleges of the United States; the study was under the direction of Professor F. Mosteller of Harvard University. He divided the colleges and universities into five classes, according to the size of the student body; the first class consisted of those having fewer than 500 students, the fifth consisted of those with more than 7000 students. He also grouped the new courses into five categories, namely, algebra (of modern type, for example, linear algebra), probability and statistics, analysis, geometry and topology, and logic For each of these categories he estimated the percentage of students in the colleges of each class to whom at least one course of that category is available. It is not surprising that, with only one exception, in each category the availability increases with the size of the college; it is disappointing that the difference is so great. Forty years ago, modern algebra began to appear in the colleges of some universities, and it has finally infiltrated even into the small colleges. Forty-one per cent of the students in the colleges of the first class can take courses in modern algebra, while in the fifth class (the universities of more than 7000 students), 95 per cent can take such courses. This does not mean that modern algebra is not taught at the other large universities, but that the courses are not open to undergraduates. The course may be modern, but the limitation is old-fashioned.

In the other categories the situation is worse. I shall state two numbers for each category: the first is the percentage of students in colleges of the first class (the small ones) to whom a course in that category is available; the second is the corresponding number for universities of the fifth class. Here are the numbers: In probability and statistics, from 14 per cent to 54 per cent; in geometry and topology, from 11 per cent to 53 per cent; in analysis, from 4 per cent to 54 per cent; in logic, from 0 per cent to 62 per cent.

Thus we come to a new aspect of the problem. Up to now I have been speaking from the point of view of the university, in which we can count on faculty members who already know what modern mathematics is. The problem is to reorganize the courses in such a way that the students will learn as much as possible in the time available. In many small colleges, there has been loss of contact with the stream of living mathematics. Before teaching modern mathematics to the students, the professors themselves have to learn it. First we must awaken them. Often they do not realize that the mathematics which they learned in their student days (and which they have partly forgotten) is not the best that can be taught today. This is not a problem with a simple solution. Perhaps no solution exists at all, and the best we can hope for is some improvement. One aspect of the improvement is the system of visiting lecturers of the Mathematical Association of America. These are well-known mathematicians, who for some months submit themselves to a severe regimen of visits to colleges, in order to give lectures on mathematical subjects. In part, these lectures are supposed to show the students what is going on in contemporary mathematics; in part, they show it to the professors. In any case, they constitute a contact between the professors in the small colleges and the active mathematicians. We may hope that occasionally they indicate to the professor the importance of continued study for those who wish to avoid ossification. If the only effect of the visiting mathematician were that the professors in the college went back to reading the mathematical journals, the visit would have been fully worth while.

But if some of those professors realize that they ought to go back to a university to study, it should be possible for them to do so. And in fact it is possible. There are summer institutes, supported by the National Science Foundation, in which they can learn something about what has gone on since they were students. It is true that there are not nearly as many such institutes as there are institutes for secondary school teachers. It is to be hoped that there will be more of them in the future. In any case, those that do exist have not failed to produce an effect. They have stirred up the corps of professors so as to facilitate the diffusion of new ideas everywhere.

We university professors should always have recognized that the advances of mathematics can be reflected in the content of elementary courses. It has always been our duty to take account of possible improvements in the teaching of mathematics. For many years we have not paid attention to that duty. Nowadays we are no longer ignoring it. But we cannot solve this problem and return contentedly to the pleasure of research. It is not a problem; it is a lasting duty. And not only is it lasting, it is growing. Every year sees the publication of a flood of papers on mathematics, and the flood increases faster and faster. Some day we shall know whether this is a healthy development, or whether mathematics will die of obesity, strangled in its own fat. Certainly the extrapolation to the year 2000 frightens me. But even in that year, if there are people and if some of them are in universities, it will still be the duty of mathematicians to reconsider the courses in mathematics, so as to make the best possible use of the time available. We should deprive those who follow us neither of the usefulness of applied mathematics nor of the enjoyment of pure mathematics. I hope that I have told you something of the present and local state of a long-lasting task.

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Discussion Following Addresses of Professors Torres (Mexico) and McShane (U.S.A.)

The discussion period following these two addresses on "New Ideas in University Instruction" was short and centered around the need for coming to grips with the problem of revitalizing the curriculum in mathematics at the highest level. The addresses of Professors Torres (Mexico) and McShane (U.S.A.) found support in the following statements.

Prof. Balanzat (Venezuela): Professor Torres' address has directed attention to the main problem involved in modernizing the teaching of mathematics in this part of the Americas. It is all of fifteen years since France modernized its university curriculum in mathematics but she has not yet been able to do the same for the secondary school. I believe, however, that we should concentrate our strength in modernizing our teaching of mathematics at the highest levels of instruction and decide definitely on what of the old must be eliminated and what is to be retained. For example, at the secondary school level, it would be ridiculous to introduce topology, but nevertheless we must eliminate the "fossils." We must arrive at some agreement as to the type of modern mathematics that must be taught in the three-year training period for those who will become teachers of the subject. This should be one of the basic outcomes of this conference and the most important of all. Even though in Europe, in 1960, an agreement of this type was reached, we must study our own situation because our course of study for the baccalaureate differs from the European. Although we will have to keep a great amount of the old mathematics, it would be absurd, for example, to teach the system of rational numbers by the old method when we can treat them as imbedded in an integral domain of an ordered field, which would involve the same amount of work. For this reason, I ask that we settle on the topics that should be thrown out, those that should be kept, and those that should be added.

Prof. Fehr (U.S.A.): In accordance with the remarks of Professor Balanzat, I should like to suggest that Professor Choquet explain a similar program which European universities have accepted.

Prof. Choquet (France): I am happy to have this opportunity to talk to you about a meeting that was held in Paris last year, under the direction of Henri Cartan, and sponsored by the Federation of Teachers, with such countries as Germany, Belgium, Switzerland, and Italy taking part. This conference had as its goal the adoption of a minimum program for the first three years of the university so that the transfer of a student from a university in one country to a university in another would be facilitated as in medieval times. The interesting thing is that agreement was reached so quickly that I have the results with me in printed form and these will be translated [and distributed] immediately.

The program is divided into two parts, a first level and a second level. The first level, intended for the first two years and called "preparatory," consists of the study of algebraic and topological sets. The subjects for the second level are those which in France used to be designated as "differential and integral calculus," but with an entirely modern accent and including topics necessary in the preparation of students for teaching in the secondary schools of the lycée type. I end my remarks, Professors Torres and Balanzat, with a plea for a similar effort on this continent, mindful, naturally, of the differences in educational traditions.

*THE MATHEMATICS PROGRAM IN THE SWISS SECONDARY SCHOOLS**

An Experiment in the Gymnasium of Neuchâtel

Professor Laurent Pauli (Switzerland)

Before considering the main topic of this discourse, it is necessary to recall the fact that Switzerland is a confederation of twenty-five states. Each of these states is sovereign in matters concerning education, which means that in Switzerland there are twenty-five secretaries of education and no central department. Nevertheless, there is a federal policy governing "maturity examinations," but this policy is so vaguely formulated that it gives the cantons complete liberty in setting the examinations and administering them.

Rather than speak about the teaching of mathematics in all twenty-five cantons, I prefer to select a sample program, in this instance, that of Neuchâtel, where the mathematics curriculum has been radically modified during these last few years. As a teacher of mathematics at the Gymnasium of Neuchâtel since 1937 and as director of the school since 1946, I introduced, during the period 1946-1959, modern methods into the teaching of mathematics without modifying the general program. During the winter of 1959-60, I embarked on a complete reorganization of the program. In order to provide a background for understanding the scope and significance of the experiment I am about to describe, a few points must be clarified.

The Gymnasium of Neuchâtel receives students at the age of 15 years and prepares them to take the maturity examinations after three and a half years of study. The school has three sections: a literary section, with Latin required; a scientific section; and a pedagogical section. The students in the literary section will become theologians, lawyers, doctors of medicine, teachers of languages or of sciences, and even, in some cases, engineers. The greater part of the mathematicians, physicians, and engineers are recruited from among the students of the scientific section. As for the students in the pedagogical section, they will, in general, become school teachers, that is, elementary school teachers. The students entering the Gymnasium know the fundamentals of the algebraic processes and the solution of first degree equations with one or several unknown

*Address translated from the French.

quantities. They have been taught plane geometry either intuitively or semi-deductively. By this is meant that, beginning with a certain number of facts established experimentally, they learn how to demonstrate several fundamental theorems, for instance, the theorems of Thales and of Pythagoras. The process of modernizing the teaching of mathematics has taken place in two stages. In the spring of 1960 we set up, in the scientific section, two pilot classes that will take the maturity examinations in 1963; in 1961 the new program for mathematics was given in all the classes of the first year of the scientific section as well as in the two pilot classes of the second year. Moreover, all the classes in the first year of the literary and pedagogical sections (classes of 15-year-old students) received, beginning in April, 1961, a modern course based on an absolutely new curriculum.

PROGRAM OF THE SCIENTIFIC SECTION

Let us take note of the fact that among the students in this section are future mathematicians and physicians, and future engineers of all fields (mechanics, electricity, civil engineering, electronics, etc.). Therefore it is important that the new program, on one hand, provide superior teaching of modern mathematics and, on the other hand, give future engineers good techniques in the fundamentals. More precisely, if the introduction of set theory, the notions of groups, rings, and fields is indispensable, it is not less necessary that the students be able to calculate efficiently and rapidly, and solve the problems of algebra, analytic geometry, and differential or integral calculus. Similarly, they should be capable of handling with ease, problems in plane geometry and the geometry of space. Here is the new program which has been in force since April, 1960.

First Year (age 15-16 years). 7 hours per week.

I. Algebra

1. Algebra of sets (elementary point of view).
2. Concept of mapping (elementary point of view).
3. Natural numbers; congruences.
4. Integers.
5. Rational numbers.
6. The existence of irrational numbers; first concept of the set of real numbers.
7. Complex numbers.
8. The concept of polynomial.

9. Numerical value; divisibility by $(x-a)$; Horner's method.
10. Problems of division; zeros of a polynomial; Viète's theorem.
11. Equalities, equations, identities.
12. Equivalence.
13. Approximate solutions (elementary); slide rules and calculating machines.
14. The equation of the 2nd degree.
15. Equations in the form of the 2nd degree.
16. Combinatorial analysis.
17. Review, practice.

II. Geometry

1. Axial symmetries.
2. Translations.
3. Rotations; central symmetries.
4. Homotheties.
5. Generalizations to space of items II, 1-4.
6. Selected problems in metric geometry.
7. Review, practice.

III. Vector Calculus

1. Vectors; operation in R_2.
2. Plane analytic affine geometry.
3. Extensions to R_3 of III, 1.
4. Extension to R_3 of III, 2.
5. Introduction of a metric into R_2; plane geometry, analytic and metric.
6. Generalization of concept of arcs and angles (congruences).
7. Trigonometric functions.
8. Synthesis of I, 7; III, 7.
9. Synthesis of I, 3-7; II, 1-4; III, 1-8.
10. Synthesis of I, 2; I, 13; III, 7.
11. Introduction of a metric into R_3.
12. Some applications of plane and spherical trigonometry involving calculations.
13. Review, practice.

As in the past, teaching is based on knowledge previously acquired, but practice will be provided from a new point of view that will inspire interest in the students. In the course of this year, teaching will be on an intuitive basis, so that no general theory will be presented but new concepts will be successively introduced by means of judiciously chosen exercises. Laboratory work with numbers as well as with geometry can be foreseen. Subjects will be arranged according to the final goals and the structures which will be made evident during the years to follow, but all formal expositions will be avoided. Teaching in the first year can be characterized by the general heading: Teaching of techniques, notations, and fundamental concepts.

Second Year (age 16-17 years). 7 hours per week.

I. Algebra (structures)

1. Groups.
2. Rings and fields.
3. Vector spaces.

II. Geometry

1. Schema; models; concept of axiomatic basis.
2. Affine geometry; affine transformations (affinities); group of affine geometry; analytical point of view: matrices.
3. Projective geometry; homographic transformations (homologies); projective group.
4. Polarity; duality.
5. Inversion.
6. A problem in metric geometry: bundles of circles; analytical point of view: affine geometry of circle and of sphere.
7. Motion as a particular case of affine transformations.
8. An axiomatic basis of metric geometry; models of Poincaré or of Klein; introduction to non-euclidean geometries.

III. Analysis

1. Construction of the set of real numbers; isomorphism of this set with the set of points of a line.
2. Denumerable set; set of rational numbers is denumerable, that of real numbers is not.
3. Generalized exponents; logarithms.
4. Domain of definition of a function; inverse function; increasing and decreasing functions.
5. Neighborhoods; limit of a function; continuity in x_0; examples of discontinuities; real value.
6. Rules of calculation of limits.

7. Derivative of a function of x_0 in a certain neighborhood.
8. Rules for the first derivatives of $y = x^n$; $y = u \pm v$;

$$y = uv; \quad y = \frac{u}{v}; \quad y[u(x)].$$

Teaching in the second year can be characterized by the following: axiomatics and deductive exposition. By review, applications, and new developments, particular structures of different disciplines will be successively presented. It will be possible to introduce the idea of axiomatic basis and axiomatization for different topics: geometry, theory of numbers, theory of groups, etc. To a certain extent, this is a matter of revealing to the students the theoretical development of topics treated intuitively in the first year.

Third Year (age 17-18 years). 5 hours per week.

A detailed program has not yet been established. The following is a general outline.

Differential calculus and integral calculus.
Taylor's formula; several expansions in a series.
Systematic study of linear mappings.
Quadratic forms (reduction to the canonical form).
Application to the study of conics.
Derivatives of a vector and application.

Commentaries on the Program

For teaching the above, it is necessary to arrange the subjects sequentially and to classify them. It should be pointed out that there are no longer any lines of demarcation separating algebra, geometry and vector calculus. Concepts of the theory of sets as well as mappings, introduced at the beginning, are used constantly. Topics in I, 3,4,5 permit the introduction of concepts of groups, rings, and fields which are found again in geometry and vector calculus. Subjects are not taught in the order given above but horizontal connections are constantly being established. In this way, after the presentation of I, 1 and I, 2, it is possible to study I, 3-7 and III, 1-5 in parallel sequences. Topics under the title "Synthesis" permit us to regroup certain concepts and to review them in a new perspective. Thus the relations which exist between complex numbers I, 7, on the one hand, and trigonometric functions III, 7, on the other hand, are pointed out. The concept of group permits us to establish connections between I, 3-5, II, 1-4, and III, 1-7.

We have mentioned above that teaching in the first year should be concrete and intuitive. Let us consider several samples in order to illustrate this idea. At first vectors are introduced with the help of arrows. Students learn how to operate with the arrows, to add them, and multiply them by a scalar. They solve numerous geometrical problems involving the task of finding a vector sum. The properties of the parallelogram and those of translations form the basis for this first introduction. It is in the second year that an axiomatic structure of vector space will be presented. Also, students will experiment with cardboard figures in order to study symmetries and rotations. It is with the help of these models that they themselves discover the groups associated with the equilateral triangle and the group for the square. Relations between square and rhombus, square and rectangle, permit the easy introduction of the idea of subgroup. We have taught our students how to explore number relations. In this way, having divided the numbers between 1 and 98 into seven classes, for instance, the students established for themselves the table of addition of the classes and also of multiplication. Their calculations showed them that every equation of the form $A + X = B$, where A, X, and B designate the residue classes, modulo 7 always has one solution. The same is true for equations $AX = B$ and $XA = B$. If they compare this result with those that they obtain in the set of natural numbers, they easily find the differences. These findings lead naturally to concepts of groups, rings, and fields. A systematic analysis of cases where the modulo is any number n or a prime number p allows for generalization of the discoveries made experimentally.

Future engineers should be capable of performing numerical calculations in a rapid and sure way. For this reason, we have systematically introduced calculation with the slide rule. The numerical solution of a triangle is done immediately with the help of formulae:

$$\frac{a}{1} = \frac{\sin\alpha}{\frac{1}{c}} = \frac{\operatorname{tg}\alpha}{\frac{1}{b}}$$

where $\alpha \leqq 45°$, and c is the hypotenuse.

In the same way, the law of sines, directly or with the help of two or three applications, leads to the solution of any triangle. After the study of topic I, 9 our students make use of calculating machines. In this way, they learn to solve, by approximation, equations of any degree, just as later in the third year they will solve transcendental equations.

Let us point out again the fact that it has seemed very important to us to introduce the concept of the theory of sets in easy stages, and not in one unit, at the beginning of teaching.

PROGRAM OF THE LITERARY SECTION

The success of experiments conducted during the year 1960 in the scientific section encouraged us, as we mentioned above, to introduce a new program of mathematics in the literary section in the spring of 1961. As we have only 4 hours per week at our disposal, this was not a matter of merely adopting the program of the scientific section. It was necessary to conceive of teaching as being divided into three years but forming a whole. Here is the program of the first year with the number of weeks suggested for treating the subjects.

First Year (15-16 years). 4 hours per week.

Topics

1. Sets
   Examples[1]; belonging to a set; subsets; inclusion; complementary sets. (1 week)
2. Relations
   Reflexivity; transitivity; symmetry; examples (equivalence and order). (2 weeks)
3. Mappings
   In; on; biunique; examples. (1 week)
4. Operations
   Product set; associativity and commutativity; composition of mappings. (5 weeks)
5. Groups
   Identity element; inverse; definition of a group; abelian group. (3 weeks)
6. Vectors
   Definition of vector in a plane; equivalence; addition; additive group; multiplication by a scalar; theorem of Thales; concurrence of medians. (4 weeks)
7. Analytic Geometry (affine)
   Basis and components; equations of the line (except scalar). (4 weeks)

We shall soon set up a detailed program for the second and third years. The second year will be devoted to geometry (scalar product, vector product, analytic geometry with 2 and 3 dimensions,

[1] Unless otherwise stated, "Examples" will mean the following: concrete examples; numerical, algebraic, or geometrical (the last three drawn from material already known to the students).

trigonometry). The third year will involve a more profound study of the concept of number and of algebraic structures, elements of differential calculus, and mappings.

Let us not forget that the majority of the students of this section will not pursue a scientific course of study. Therefore instruction has an essentially cultural function which accounts for the emphasis that we give to fundamental structures. It is very important that on entering the University these students have concepts of modern mathematics that are as clear as possible, so that the world in which they will be living will not be strange to them. We should not forget that in a great number of countries people with a training in literature occupy government positions. The role of the gymnasium is decisive in their development. It is ridiculous to give future lawyers, for instance, a mathematical conception which is at least two hundred years old. Therefore we follow with great interest the experiment which is being conducted in the literary section.

PROGRAM OF THE PEDAGOGICAL SECTION

In his address at the seminar of Royaumont, Professor Stone declared that every reform in teaching of mathematics should begin in the elementary school. Therefore teaching of elementary arithmetic merits our full attention. Such a reform, however, is possible only if future teachers receive a sufficient background of mathematics. This is the reason for our decision that the program of the pedagogical section should be analogous, in its general outline, to that of the literary section. (We have only three hours weekly instead of four, therefore certain topics—differential calculus, analytic geometry—have to be cut short). The fundamental operations will be taught more effectively if the teachers have received suitable training in modern mathematics, that is, if they have, in particular, clear and correct concepts of fundamental structures. They should understand the role of commutativity, associativity, distributivity. The extension from natural numbers to rational numbers, eventually to signed numbers, should also be analyzed with great care. Beginning with the first year of school, actually, it is possible to draw attention of students to the concepts of commutativity of addition and, later on, to show the same property for multiplication. To be sure, this is not a matter of using the abstract word but of illustrating the concept. Thus the senior students of our pedagogical section have introduced in their primary classes a sheet of examples with a collection of operations presented in the following manner:

$8 + ? = 17$	$8 \times ? = 56$	$0.6 \times ? = 0.54$
$? + 7 = 13$	$? \times 7 = 42$	$? \times 0.7 = 0.35$
$14 - ? = 6$	$48 \div ? = 8$	etc.
$? - 8 = 7$	$? \div 6 = 9$	
etc.	etc.	

If we reviewed the steps in calculation that we have just proposed, we would very soon be led to replacing the question marks by a letter. In this way we shall have prepared for algebraic operations, and, in particular, the solution of equations of the first degree.

On the other hand, teaching the measure of lengths, surfaces, and volumes is possible only if the teachers have studied these same concepts from an advanced point of view. They will then be able to show the students of 9 or 10 years of age, with the help of cutting and folding, the properties of the square, rectangle, rhombus, parallelogram, isosceles and equilateral triangles. It should be noted that it is not a matter of systematic teaching but of experiments performed by the children. However, within these experiments are hidden the concepts of groups related to rotations or to symmetries and only those teachers who understand these notions will know how to profit by this manual work of the students.

CONCLUSIONS

It is at the same time both easy and difficult to evaluate the experiments which I direct and in which I participate as teacher of mathematics. In fact, I know the inside work: day after day the reactions of students have made us modify our methods, and change our points of view. But how is it possible to give an objective judgment of an activity in which one is completely involved?

I would not like to give you an impression that I blindly praise the programs and methods used in the Gymnasium of Neuchâtel. I am better qualified than anybody else to gauge weak and strong points; I am probably the most severe critic of the experiments which are going on in our classes.

I must add that these experiments have been made possible, on the one hand, by virtue of the training given to me by my teachers of mathematics, H. Hopf, G. Polya, F. Gonseth, in the École Polytechnique Fédérale de Zürich, and on the other hand, by virtue of the team spirit and good will of my colleagues in the teaching of mathematics. Today, in their absence, I want to pay my respects to the disinterested efforts that these teachers have given and are still giving.

Actually, in order to introduce the new program, we have organized weekly seminars for the teachers of mathematics. Their individual scientific training has made it possible for them to adapt themselves without difficulty to the new program. We have seminars of two or four hours every two weeks and at these we discuss the results of the current experiences.

It is possible to establish a provisional balance sheet:

1. The students have reacted positively and with a great interest. On the whole the school results are somewhat superior to the results of the preceding years.

2. The new concepts should be introduced at the moment when they are necessary. This remark is especially important for the concepts of the theory of sets.

3. It is important that the students keep up their knowledge of numerical and algebraic calculation; that they continue practicing the fundamental constructions of geometry.

4. It is probable that in a more elementary form the subjects that we introduce at 15 years could be introduced three or four years earlier. This is what is proposed in the program of O.E.C.D.

5. We should take care not to publish the manuals too soon. Mimeographed materials that we have used can be constantly revised and modified as the course proceeds. Only at the end of one cycle of the experiment can we hope to publish manuals.

Our school is willing to welcome all who are interested in our work. Moreover, we shall be happy to establish and to maintain regular correspondence with other schools trying similar experiments. In Switzerland we are in regular contact with our colleagues in Lausanne and Geneva who have introduced a program of modern mathematics into their classes.

In a world which progresses so rapidly, it is no longer possible to work in the closed circle. Exchanges and conferences are indispensable. As far as we are concerned, we are anxious to have them.

* * * * *

Discussion Following Address by Professor Pauli (Switzerland)

The account by Professor Pauli of an experiment in modernizing the mathematics curriculum in the Gymnasium at Neuchâtel raised many questions about the program which were addressed to the following topics: (1) The relation between instruction in calculus and in physics. (2) A comparison between the ages of the students in the gymnasium of Switzerland and the lycée of France. (3) The effect of training in "modern" mathematics on the performance of the student in other subjects. (4) Availability of textbook materials. (5) Cost of calculating machines used at Neuchâtel.

Professor Pauli's response (abridged) to these questions is given below:

1. In Switzerland, the concepts of the differential and integral calculus are presented in two stages, in a manner comparable to that used in presenting the notion of vector space. This is done with constant reference to physics, since the teachers require it.

2. The ages of students in the gymnasia of Switzerland vary with the canton. At Neuchâtel, students graduate at 18 years of age, whereas, for most of Switzerland, the age is 19. (Professor Schwartz (France) remarked that this is considerably older than the age of the French graduate.)

3. Training in modern algebra, set theory, etc., has had a spectacular effect on the performance of the student in courses in philosophy. Students have been asking unusual questions in these courses.

4. There are no textbooks that meet the requirements of the new course at Neuchâtel. All materials are multigraphed. This has the advantage of flexibility in that changes are easily made.

5. Every possible effort is made to keep expenses for the student at a minimum. The school therefore keeps on hand about 200 slide rules which may be borrowed by the students. The school also has about fifty calculating machines which are sufficient for class work. One of these machines costs about 400 Swiss francs.

THE MATHEMATICS PROGRAM IN DENMARK

Professor Sven Bundgaard (Denmark)

In Denmark, the need for a modernization of the mathematics curricula and the possibility of obtaining reforms have been studied and discussed intensively since about 1956. Considerations have resulted in changes in the introductory university instruction, and in new mathematics curricula for secondary schools that were approved in 1960 by the central governmental school authorities. The changes are more substantial at the upper secondary level (gymnasium, grades 10-12, ages 16-19) than at the lower secondary level (realskole, grades 8-9, ages 14-15).

An exact syllabus of the subject matter constituting the official curriculum for the so-called mathematics-physics branch of the gymnasium (grades 10-12) is furnished in Section II of this report. Section I deals with the background for the reform work, including teacher training, with some reference to the lower secondary level (grades 8-9).

In all that follows, it should be remembered that Denmark is a very small country with only four and one-half million inhabitants concentrated in an area of approximately 45,000 square kilometers.

SECTION I

Developments since 1900

Until recently, mathematics instruction in Denmark has followed rather traditional lines. However, a few remarks pointing to special circumstances are perhaps not out of order.

The influence of the work of Felix Klein on problems connected with mathematical instruction reached Denmark shortly after 1900. In geometry, textbooks appeared for the lower secondary level (grades 8-9), with a setup rather far from that of Euclid. In the mathematics-science branch of the gymnasium (grades 10-12), in accordance with a school act in 1905 calculus was introduced to a substantial extent; complicated euclidean geometry and manipulative algebra were reduced considerably. Two Danish mathematicians, T. Bonnesen and J. Hjelmslev, who had close connections with the Felix Klein group, were instrumental in this movement. Somewhat later, new and extremely good textbooks by H. Bohr and J. Mollerup, written for instruction in analysis at the Institute of Technology and used for the introductory course in the university, exerted a strong influence on the teaching of calculus in the gymnasium. Around 1930, this teaching reached a rather high level in many schools. Manipulative skill and a careful introduction to the basic notions of limit, continuity, derivative, and rigorous proof were made goals of this instruction. It has been said that $\epsilon - \delta$ considerations conquered instruction in mathematics at the Danish gymnasium.

From about 1935 to the present time, practically no changes have taken place in secondary mathematics instruction.

The textbook by Bohr and Mollerup, Laerebog i Matematisk Analyse, made its appearance in 1920 and was used with great profit in the first two years of university instruction in analysis. This book, written in masterly style and in a spirit very similar to that of Hardy's Pure Mathematics, was a modern one at that time. Later editions have changed part of the content but not the nature of the setup as a whole. It was the basic textbook until recently. In geometry, a new introductory university textbook by B. Jessen appeared about 1940. It based differential geometry firmly on theorems proved in analysis. Vectors were used throughout. In analytic geometry, the exposition may be characterized as semi-modern; the notion of an abstract vector space was not introduced. Recent developments in mathematics were dealt with in the third, fourth, and fifth years of university studies.

One reason for the rather slow development in the introductory university instruction is that, according to tradition, the same textbooks were used at the university and the Institute of Technology. In fact, until around 1942, university students were instructed in analysis at the Institute of Technology during their first two years of study.

The Danish Commission on Mathematical Instruction

In the years following World War II, a small and slowly increasing number of interested people realized that the rich growth of mathematical ideas during our century could and should be taken into serious consideration on all levels of mathematical instruction.

In 1955 a Danish Commission on Mathematical Instruction was established. The membership was composed of mathematicians, representatives of school authorities, and members of associations of mathematics teachers. This commission succeeded in becoming instrumental in a speedy development of the new program. It received governmental economic support for its activity, and it was able to establish study groups, start discussions, and arrange meetings and summer schools for teachers.

Acting as the Danish subcommission of the International Commission on Mathematics Instruction (I.C.M.I.), it could supply the national reform work with adequate information about similar efforts in other countries. Such information has played an important role, and international meetings such as the Royaumont seminar have had a most stimulating influence.

Effects of a New Danish School Act

One particular event has exercised a very strong influence on the development in Denmark. In 1958 the Danish parliament passed a new school act. Simultaneously it was decided to undertake a revision of all curricula of the various subjects and grades. Relatively short time was allowed for the revisions. Instruction according to the new plans was scheduled to start in August, 1963, and for some grades even earlier. The new curricula had to be worked out as quickly as possible in order that corresponding textbooks could be made available. In fact, the new curricula were approved by the authorities as early as 1960. This general situation offered a unique opportunity for taking a substantial step toward a modernization of Danish school curricula in mathematics. It was decided to try to take advantage of this opportunity in spite of obvious drawbacks connected with the narrow time limit. One such drawback has been the insufficient time left for an extended and systematic study of this important problem: "Which subjects in modern mathematics are suited to programs of secondary school instruction, and to what extent should general, important notions and contemporary mathematical language enter school instruction?" A more limited question was unavoidable: "What parts of modern (or relatively recent) mathematics have gained such importance within mathematics itself or in applications that they should no longer be neglected in secondary school instruction?" Thus, it was necessary to deal with what should be taught rather than what could be taught. Another drawback was the insufficient time left for a genuine preparation of the teachers and for experimentation with textbooks. Lectures on relevant topics and discussions at meetings for teachers have taken place on only a modest scale.

The Danish School System

It may be desirable for the reader to know a few facts about the new Danish school system.

After seven years of elementary school, a large part of the Danish pupils are transferred to the three-year realskole. Having attended this school for two to three years, a rather small number (at present 6-7 per cent of all children) are admitted, on the basis of qualifications, to the three-year gymnasium (grades 10-12). About half of this group will attend the mathematics-physics branch.

In the elementary school, arithmetic is of course taught intensively; little mathematics is taught in grades 6-7. In the realskole, mathematics is given more attention; an integration with arithmetic is aimed at. In the mathematics-physics branch of the gymnasium, mathematics is taught in 5-6-6 fifty-minute lessons per week throughout the three years; physics-chemistry is taught in 6-6-6 fifty-minute lessons per week and the rest of the time is devoted to geography-biology, foreign languages, and the humanities. Written examinations at the end of the gymnasium are the same all over the country. A commission, established by the central school authorities, formulates the problems. There are other branches of the gymnasium with fewer hours in mathematics.

Training and Retraining of Teachers at the Lower Secondary Level

Any improvement in instruction depends strongly on the quality of the teachers, their knowledge, and the spirit which they bring to their subject. Thus the basic training of teachers, as well as their opportunities for retraining, are matters of importance.

The teachers of the Danish elementary school and realskole have received 3-4-year basic training given at about thirty teacher-training colleges spread over the country. At a Danish teacher-training college, the students are taught a wide variety of subjects. For prospective mathematics teachers, there is some emphasis on mathematics. The curriculum is more limited than that of the mathematics-physics branch of the gymnasium (cf. Section II). Further, mathematics teachers giving instruction at the teacher-training colleges command, in many cases, a very modest mathematical knowledge; the majority are not university educated.

In my opinion, the basic education of teachers in mathematics at the lower secondary level (grades 8-9) is clearly insufficient. In physics, the state of affairs is similar, perhaps worse. Among responsible people there is a growing understanding of this serious weakness in the Danish educational system, and recently, the Danish Ministry of Education has decided to make a careful study of the situation. It has already ruled that future teachers at teacher-training colleges should be required to have a 5-6-year university education.

A central institute, called Danmarks Laerehøjskole, has permanent charge of the retraining of teachers in the elementary school and the lower level secondary school. Until recently the staff of the mathematics department was insufficient and partly incompetent. It has, however, been reorganized and an increased and improved staff consisting of people with a good knowledge of contemporary mathematics is preparing a more satisfactory program, including courses and summer institutes, where modern viewpoints will be presented and discussed. Jointly with the universities, the central institute is also planning the retraining of the teachers at the teacher-training colleges.

Training of the Gymnasium Teachers

In principle, the basic training of mathematics (and physics) teachers at the gymnasium (upper secondary level, grades 10-12) is much more satisfactory. It consists of a university education with five to six years devoted exclusively to the study of mathematics-physics. After about three years of undergraduate study, each student has to choose a rather narrow field for two to three years of graduate study. In continuing university study, prospective gymnasium teachers are required to receive training in actual classroom teaching for 12 hours per week during half a year; during the same half year they are required to attend lectures on pedagogical subjects and study literature on the history of education and psychology.

Thus far there has been no permanent retraining system for gymnasium teachers. In view of the substantial basic education, the teachers are supposed to continue individual studies simultaneously with their teaching, keeping themselves informed about developments in their subject. To a certain extent, this has worked out satisfactorily in subjects with a slow development. It has now been realized, however, that in fast-moving subjects such as mathematics and physics, it is no longer realistic to assume that the teacher is able to follow their development. In addition, at least temporarily, because of the severe shortage of university-educated mathematicians and physicists, the teachers are overloaded with teaching duties, with little or no time for individual studies. A permanent retraining system, including sabbatical years, is under consideration. Until the system has been established, ad hoc summer schools and meetings are being arranged.

Preparation of Gymnasium Teachers for the New Mathematics Program

A new program will hardly be successful or even approved by school authorities without cooperation and understanding on the part of the teachers in question. Problems connected with this fact are not trivial and have to be very carefully handled.

As late as 1957, only a few mathematics teachers at the Danish gymnasia agreed upon or even understood the need for changes; school mathematics had become and remained for twenty-five years a well-defined and static field. The quality of teaching and some of the textbooks had reached a rather high level.

In cooperation with the Mathematics Teachers Association, the Danish Commission on Mathematical Instruction arranged for a 2-week summer school in 1959, devoted to background material for reforms. Invitations with appropriate information, sent out as early as 1958, created considerable interest, and about 30 per cent of the teachers applied for participation. The Danish Government made grants which were sufficient to pay tuition and living costs for all the applicants. Simultaneously, the O.E.C.D. granted money for an all-Scandinavian summer school of the same type, and the two schools were joined into one. This joint school was located at the Mathematical Institute of the University of Aarhus and I had the privilege of being responsible for the arrangements. There were 120 participants. In 1960 a similar O.E.C.D.-supported summer school for mathematics teachers was held again at Aarhus.

The program emphasized (a) recent notions which have become important in mathematics, and (b) elementary probability and statistics. The schedule for each day consisted of one double-lecture for the whole group and two hours of exercises dealing with the material of the preceding day's lectures. The participants were supposed to devote several hours a day to study-work in order to become really familiar with the material. They were also required to solve problems given as exercises in the lectures and to present their solutions. Seminars were held for small groups of participants.

The main emphasis was on item (a) comprising, of course, such concepts as sets, relations, mappings, elements of topology, including metric space, binary composition including homomorphisms, and topological groups. However, if there was a special feature to the instruction, then it was the choice of examples, comments, and problems. We considered it most important to convince the gymnasium teachers that what we presented to them had very much to do with important things with which they were used to dealing. Furthermore, we aimed at demonstrating that it was not only possible but rather easy and profitable, to use general notions and reasoning based upon them, in this way treating simultaneously details from different fields which otherwise seemed to have little or no connection. As basis for the instruction, lecture notes had been prepared in two volumes, one containing definitions and theorems and the other, examples, comments, and problems. The two volumes, a total of about 300 pages, include more material than was actually dealt with at the summer schools; it was intended that the participants should be stimulated to continue studies afterwards.

The large majority of the participants showed great interest. They worked very hard and with good results in spite of the great amount of material presented in so short a time. In view of the reasonably good basic training of the participants, we found it appropriate to extend the work beyond definitions and their immediate consequences. Supplementary lectures were offered on such topics as contractions in a metric space with application to existence theorems for differential equations, and the Stone-Weierstrass theorem.

At both of the summer schools, a preliminary draft of the new curriculum was presented to the participants and discussed with them. Undoubtedly this activity prepared in a very effective way for the coming reforms. About half the gymnasium teachers of mathematics, as a whole the best and most active, attended one or both of the summer schools. Many who did not attend the summer schools studied the lecture notes.

The final draft of the new curriculum was presented at a general meeting of all gymnasium teachers in mathematics some time after the second summer school. The new program was agreed upon by a considerable majority. As mentioned before, it was finally approved by the central governmental school authorities and the Ministry of Education.

Various groups, in each case comprising both university mathematicians and teachers, are writing textbooks which are published on a free enterprise basis.

Shortage of Teachers. Working Conditions

Upon its establishment in 1955, the Danish Commission on Mathematical Instruction immediately turned to the government, explaining in detail that a strongly increasing demand for university-educated people in mathematics-physics was to be expected, and that a shortage of teachers, already being felt at that time, was likely to increase rapidly. In a memorandum to the Minister of Education, the Commission estimated that the number of university students in mathematics-physics had to be multiplied by a factor of 4 in order to fill existing and future gaps. Among other things, it was pointed out that the teaching profession had to be made more attractive.

This action was strongly supported by influential people, in industry, for example, with reasonable understanding of the importance for technology of the basic sciences. The action was publicly discussed, and, further, dealt with in a Committee on Education of Technical and Scientific Personnel, charged by the Government with working out suggestions as quickly as possible. A large-scale expansion of the Technical University, of the mathematics and physics departments of the Universities, and of the Danmarks Laerehøjskole was immediately decided upon, and appropriate funds for new buildings (corresponding to about U.S. $200 million) were made available. In addition, perhaps as another consequence, staff salaries were considerably increased.

This action was followed up by appropriate information, distributed through radio and newspapers and also sent directly to pupils and teachers. One effect has been that, in 1961, more than seven times as many students applied for university studies in mathematics-physics as in 1955. There is no evidence for believing that the quality of the students has decreased.

However, as predicted in 1955, the shortage of teachers temporarily became very severe. The "normal" teaching load for a gymnasium teacher is 24 hours per week, including hours for correcting written solutions to problems; the number of actual lessons is about 18 per week. But at present most teachers have at least 30 lessons per week, some even much more. The "normal" salary at the first appointment of a gymnasium teacher is equivalent (according to buying power, not official exchange rate) to about U.S. $6,000 per year; it increases according to seniority to about U.S. $9,000. Extra hours are well paid. It is rather common that the salary of a young teacher amounts to the equivalent of about U.S. $10,000, and that of an older teacher, to about U.S. $15,000 per year. Cases of an income equivalent to U.S. $25,000 are publicly known and have even been dealt with in cartoons. Thus, at the present time, it is by means of the severe overworking of the mathematics teachers that it is possible to keep the Danish gymnasia running.

Since the teachers have very little time left from routine work, this might very well contribute to a delay or reduction in the success of the new mathematics curriculum.

The New Mathematics Curriculum at the Lower Secondary Level (Grades 8-9)

It was noted earlier that the basic training of mathematics teachers at the lower secondary level (8-9) as well as their retraining is unsatisfactory. When preparing the new curriculum it was considered necessary to take this fact into account. Only a very modest modernization of teaching has been made compulsory. On the other hand, freedom is left to the teachers to go far in modernizing their teaching, according to their knowledge, and the curriculum is accompanied by recommendations stimulating new points of view. Thus, future retraining activity will have a good chance of influencing teaching in coming years. Textbooks written from a modern point of view can be used as well as traditional books.

The New Mathematics Curriculum at the Upper Secondary Level (Grades 10-12)

According to a Danish tradition, much freedom is, in principle, left to the teachers to arrange their instruction according to their taste and judgment. Therefore the official curriculum is merely a list of topics, accompanied by official recommendations and comments by central school authorities.

Section II of this report contains the exact translation of the subject list of the new curriculum for the 3-year mathematics-physics branch of the gymnasium. Some remarks concerning the official recommendations are added in order to provide the reader with a more detailed picture. In other branches of the gymnasium a certain degree of modernization following the same trends has been introduced.

Responsible people are quite aware that the new curriculum in mathematics represents a modest progress. In coming years, the problem will need to be given careful study, and further possibilities for bringing school instruction into reasonable accord with the development of mathematical ideas will have to be considered in preparation for future reforms.

SECTION II

Mathematics Curriculum of the Mathematics-Physics Course in the Danish Gymnasium

1. General concepts from set theory and algebra. Set, subset, complement, union, intersection; equivalence relation, ordering relation; mapping of a set into and onto another (the concept of function), one-to-one mapping, inverse mapping (inverse function); denumerability; binary composition: group, ring, field.

2. Numbers. The natural numbers; axiom of induction; primes; greatest common divisor; Euclid's algorithm; the ring of integers, equivalence classes modulo an integer; the field of rationals, its denumerability; the field of real numbers, its continuity, upper and lower bound, non-denumerability, infinite decimal fractions, absolute value; the field of the complex numbers.

3. Combinatorics. Combinations and permutations, binomial formula; the concept of a finite probability distribution; examples of determination of probabilities by means of combinatorics.

4. Equations and inequalities. Equations and inequalities of first and second degree with one unknown; equations and inequalities of first degree with two unknowns; simple examples of other equa-

tions; second order equation and the binomial equation in the complex field.

5. Plane geometry. The rectangular coordinate system, change of coordinates; vectors and their coordinates, vector algebra including scalar product; analytic representations of a line: distance and angle; analytic representations of a circle; area of triangle and parallelogram; definition and analytic representation of parabola, ellipse, hyperbola; mappings of the plane onto itself, parallel displacement, rotation, reflection, multiplication and composition of these mappings, orthogonal affinity.

6. Solid geometry. The rectangular coordinate system; vectors and their coordinates; vector algebra including scalar product; parametric representation of a line; analytic representations of the plane: distance and angle; equation of the sphere, spherical coordinates, spherical distance between two points (the cosine relation); polyhedra, Euler's theorem, the regular polyhedra; volume of prism, pyramid, right circular cylinder and cone, sphere; surface area of right circular cylinder and cone, sphere, area of spherical triangle; congruence and symmetry.

7. Elementary functions. The linear function of one variable, and of two variables; polynomials in one variable, including their factorization; greatest number of roots, determination of rational roots in polynomials with integral coefficients; rational functions of one variable; logarithmic functions, logarithmic scale, use of slide rule and logarithm tables; exponential functions, power functions; the trigonometric functions, addition formulae, logarithmic formulae; application of trigonometric functions to oscillations and to computation of unknown sides and angles in a triangle; the linear function of one complex variable and its geometrical interpretation.

8. Calculus. The concept of limit; continuity and differentiability of a real function of one real variable; continuity and differentiability of a vector function of one real variable (tangent vector); rules for differentiation; Taylor's formula (approximating polynomials), differentials; the definite integral as a limit of sums; the indefinite integral; rules for integration, including integration by parts and integration by substitution.

9. Application of the calculus. Determination of the range of a function and intervals of monotonicity; simple examples of determination of asymptotic properties of functions; drawing of graphs of given functions, and drawing of curves determined by a simple parametric representation; velocity vector, speed, acceleration vector; determination of areas and volumes by integration; examples of applications of the calculus in probability theory; examples of applications of the calculus to numerical problems and to problems

in physics and other subjects; examples of simple differential equations.

10. A subject chosen by the teacher.

Comments on the Topics Listed Above

General remarks. The subject list is not intended to indicate a chronological arrangement of the material. It is left to the teacher to decide what he finds appropriate from a pedagogical and systematic point of view. Also, integration with instruction in physics and other subjects has to be taken into consideration.

The teachers are encouraged to comment on the origin of important notions and their historical development.

Numerical calculations are not neglected. Slide rule, tables, nomograms, and graph paper are used as tools.

When dealing with examples, e.g., from physics, the teacher is supposed to show the students that the use of mathematics is in accordance with definitions and theorems, even if the language of physics appears shorter than is customary in mathematics.

Re point 1. Set theory is employed primarily as a means of clarifying the fundamental concepts and reasonings, and as a basis for a precise and up-to-date mathematical mode of expression. The set theoretical concepts should be defined at moments when the discussion of other subjects makes this natural. General set theoretical concepts should be illustrated by various examples, both new ones and examples from material taught in grades 8-9.

Elements of logic are not mentioned in the list, but it is recommended that set-theoretical considerations be used to illustrate some of the basic logical rules. If this is done, e.g., in connection with equations and inequalities, the pupils will gain a better understanding and not only manipulative skill.

A function should be defined in the general form, as a mapping from one set into another. The use of this notion of a function throughout in the teaching of the various topics in the mathematics curriculum will have a great unifying influence.

An extended course in abstract algebra is not intended. The fundamental concepts, rule of composition, group, ring, and field form the basis for a description of the algebraic structure of the number system. Through examples from different fields of mathematics, the algebraic concepts can illuminate the connection between subjects which otherwise seem wide apart.

Re point 2. The students should realize the fundamental role played by the axiom of induction; in particular, its application to proof by induction should be made clear.

Concerning prime numbers, only the theorem that the set of prime numbers is infinite, and the theorem about unique factorization in primes are required.

A precise description of the algebraic structure, ordering, and continuity of the system of real numbers should be given, but no construction of the system of real numbers from the system of rational numbers should be included.

The students should acquire skill in working with absolute value. In exercises, absolute value should appear in equations, inequalities, and in connection with functions.

Re point 3. The students will have to become acquainted with the general concept of a finite probability field. The treatment should not be restricted to fields where all points have the same probability. This involves the presentation of a simple mathematical model and the application of the terminology attached to this model.

Re point 4. Types of equations which should be given are: A system of three linear equations with three unknowns; a system of two equations with two unknowns, one equation of first degree, the other of second degree; equations in which the unknown appears under a square root; exponential equations and trigonometric equations with one unknown. The examples of these types of equations should be simple.

Re point 5. Change of coordinates must comprise parallel displacement and rotation of the coordinate system.

In analytic geometry, the vector concept should play a central part.

The expression "analytic representation of a line" refers to coordinate- as well as vector-equations, parametric representation, and normalized equations.

The treatment of conics can be restricted to the derivation of the equations of parabola, ellipse, and hyperbola based on a geometrical definition of these curves. Special properties of conic sections (theorems about tangents, etc.) can be dealt with in exercises.

As indicated in the list of subjects, the mappings in question (parallel displacements, rotations, etc.) are meant to be considered as mappings of the entire plane onto itself and not as mappings only of single figures. However, an account should be given of how characteristics of the figures, e.g., the sizes of distances, angles and areas, are transformed under the mappings (invariant characteristics). The fact that a circle is mapped onto an ellipse by an orthogonal affinity should be demonstrated. In this connection the parametric representation of the ellipse can be mentioned.

Re point 6. Fundamental theorems about parallelism and orthogonality of lines and planes are presented (without proofs) to an extent necessary for the introduction of orthogonal coordinates and the treatment of polyhedra.

The analytic representations of planes must include coordinate- as well as vector-equations and also normalized equations.

Exercises in applications of spherical coordinates should include problems concerning geographical and astronomical subjects.

Regular polyhedra should be considered in detail only in case of the tetrahedron, the cube, and the octahedron.

Re point 7. In connection with linear functions of two variables and their graphs, one can treat problems about maxima and minima of such functions restricted to domains determined by linear inequalities (linear programming).

Besides the function-theoretical description of polynomials, it must be demonstrated that the set of polynomials forms a ring, and the analogy between integers and polynomials should be stressed.

Practice in the use of the slide rule should emphasize theoretical points and elementary applications. Nor should too much time be spent on technical aspects of the use of the slide rule.

Presentation of the application of trigonometric functions should include the concept that a linear combination of pure oscillations with the same period is a pure oscillation.

Re point 8. During the last thirty years the treatment of calculus in the Danish gymnasium has reached a rather high level. Not only manipulative skill, but a careful introduction of basic notions and rigorous proofs have been aims of instruction.

Implicit in the suggestions for teaching these topics is the recommendation that considerations of rigor should not be overemphasized.

In connection with the concept of limit, standard theorems about limits of sums and products, etc., should be mentioned. Proofs can be confined to one or two of these theorems.

The selection of proofs of the basic theorems on continuous functions is left to the teacher's decision.

The students should be given the opportunity to carry out approximate computations of the values of certain functions on the basis of Taylor's formula.

Re point 9. Calculations of volume by integration should comprise computation of the volume of solids of rotation and pyramids.

The use of frequency functions as a basis for determination of finite probability fields is understood to be based on a postulate which states that in every particular case there exists a function with the property that the probability distribution corresponding to an arbitrary finite division of the set of real numbers into subintervals can be determined by integration of this function over the sub-intervals. Thus the presentation can be kept inside the frame of the theory of finite probability fields, and the problems will actually be exercises in integration, formulated in the language of probability theory.

The treatment of simple differential equations may be limited to the mention of the equation

$$\frac{dy}{dx} = f(x) \quad \text{and} \quad \frac{dy}{dx} = g(y), g(y) \neq 0 .$$

Simple calculations of moments of inertia and centers of gravity can be included as examples of applications of infinitesimal calculus to physics.

Re point 10. Content, extent, and mode of treating the optional subject should be adapted in such a way that the students are not faced with more difficult problems than those arising from the other topics of mathematics.

Examples of the fields from which the optional subjects may be taken are: history of mathematics, number theory, matrices and determinants, theory of groups, set theory, Boolean algebra, differential equations, series, probability theory, statistics, theory of games, topology, projective geometry, theory of conics, non-euclidea geometry, geometry of higher dimensions, geometrical constructions descriptive geometry.

The optional subject may also be chosen in connection with the corresponding part of the physics course. As examples of suitable subjects, the following may be mentioned: probability theory and kinetic theory of gases, differential equations and oscillatory circuits. Finally, the optional subject may be organized in connection with subjects other than physics, e.g., probability theory and heredity.

The program for the optional subject must be submitted to the inspector of schools for approval.

THE REFORM OF MATHEMATICS EDUCATION IN THE UNITED STATES OF AMERICA

Professor Edward G. Begle (U.S.A.)

Let us start by asking why reform in mathematics education was thought necessary in the United States. One reason was that the mathematical profession itself was dissatisfied. Teachers at all levels complained that the students they received from lower levels were poorly prepared. Science teachers complained that their students could not use effectively the mathematics they were supposed to have learned. A closer look at the situation persuaded many mathematical scholars that the textbooks in use were far from satisfactory and that mathematics teachers, especially at the elementary and secondary levels, were in general poorly prepared for the teaching of mathematics.

A far more fundamental reason for the improvement of mathematics education came from the realization that the society in which we live is rapidly changing. Our society is far more dependent on science and technology and, hence, on mathematics than was the case a generation ago. Mathematics is a rapidly growing subject. New mathematics is being invented at an increasing rate and finds applications in more and more different aspects of our culture. Many important mathematical skills, techniques, and ideas, which evolved only within the last generation, are now employed in science, industry, and business. We can be sure that mathematics will continue to grow, that new applications of mathematics will be found, and that a larger percentage of our citizens will need mathematics as part of the intellectual equipment for their work.

Today we can predict neither the particular mathematical skills which will be important a generation from now nor the particular individuals among our present students who will use these skills in the future. It is, therefore, essential that we provide for all our students not only the fundamental skills which we now know to be important but also a thorough understanding of the basic mathematics which provides the foundation for these skills and will provide the foundation for the new skills of the future. Our mathematics curriculum must therefore be reformed by including materials necessary to an understanding of the basic concepts and structures of mathematics.

It is impossible to pinpoint the beginning of the reform in mathematics education which has been under way in the United States for ten years. However, one of the most important first steps was undertaken at the University of Chicago, where new text materials for grades 11 and 12 were prepared in the late forties and early fifties. These materials emphasized the basic concepts and structures and provided a strong impetus for the preparation of such materials for both the secondary schools and the colleges.

In the early 1950's, a committee of the University of Illinois undertook the preparation of a complete mathematics program for grades 9 through 12, again placing heavy emphasis on concepts and understanding as a basis for mathematical skills. In addition, this committee experimented with new methods of teaching and pointed out the importance of involving the students themselves in the development of new concepts as contrasted with older techniques, in which the students were passive recipients and only the teacher was active.

Sparked by these activities, a number of local groups in various parts of the country undertook experiments aimed at the improvement of the secondary school curriculum. Perhaps the most important of these was a group at the University of Maryland which prepared a curriculum for grades 7 and 8. The materials produced by this group also reflect a concern for the concepts and structures of mathematics which form the basis of the mathematical skills usually taught in those two grades.

While the work done by these local groups was extremely important and produced many valuable suggestions for the improvement of the mathematics curriculum for the United States, they all suffered from their local character. Each of these groups was animated by one or two individuals, as is to be expected in the case of a small group, and reflected individual idiosyncrasies of the group leaders. As a consequence, none of the recommendations or materials prepared by these groups could find widespread acceptance.

Probably the most important step in the improvement of the mathematics curriculum in the United States was the formation of the Commission on Mathematics of the College Entrance Examination Board. This board prepares and administers achievement tests in the standard high school subjects, the results of which are used by a large number of colleges and universities in connection with their selection of students for admission. The content of their examinations therefore has a strong influence on the high school curriculum. In the middle 1950's, the examiners raised the question as to the appropriateness of the mathematics achievement test they were constructing and, accordingly, a commission was formed to investigate this question.

Two important aspects of the Commission on Mathematics should be noted. In the first place its members came from many parts of the country and the Commission was, therefore, the first truly national group to concern itself with the high school mathematics curriculum. In the second place, the Commission was composed of both classroom teachers and university mathematicians working together on equal footing, and thus provided a concrete demonstration that mathematical scholars were willing and able to devote part of their time to the problems of school mathematics.

In addition to preparing recommendations for the high school mathematics curriculum, the Commission performed another important and useful function by spreading information concerning its recommendations widely about the country, and stirring up much interest on the part not only of teachers and mathematicians, but also of school administrators, parents, and others. In fact, the effective work of the Commission along these lines contributed in no small way to the success of the whole reform movement in the United States.

The flavor of the recommendations proposed by the Commission can best be seen in the following summary statements extracted from the Commission's Report:

"1. Strong preparation, both in concepts and in skills, for college mathematics at the level of calculus and analytic geometry.

2. Understanding of the nature and role of deductive reasoning in algebra as well as in geometry.

3. Appreciation of mathematical structure ("patterns"); for example, properties of natural, rational, real, and complex numbers.

4. Judicious use of unifying ideas—sets, variables, functions, and relations.

5. Treatment of inequalities along with equations.

6. Incorporation with plane geometry of some coordinate geometry, and essentials of solid geometry and space perception.

7. Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers.

8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular).

9. Recommendation of additional alternative units for grade 12: either introductory probability with statistical applications or an introduction to modern algebra."

Despite the extremely effective work done by the Commission on Mathematics, its influence was restricted in two important respects. In the first place, by its original charge, the Commission was restricted to consideration of a mathematics curriculum for college-bound students. In the United States, this means approximately the top third, in mathematics aptitude, of the entire high school population. In addition, the Commission restricted itself to a consideration of the curriculum for Grades 9 through 12, although it did make general remarks about the curriculum for earlier grades.

A second serious limitation was that the Commission, with one exception, restricted itself to making recommendations. The history of mathematics education in the United States, and attempts to reform it, show that recommendations alone often have no effect and, at best, are reflected in curriculum changes very slowly.

In one particular case, however, the Commission went beyond a mere recommendation. As a possible topic for study in the last year of high school, the Commission recommended a course in probability. Since a serious study of this topic had never before been tried in the high schools of this country, the Commission felt that its recommendations should be accompanied by some evidence of its feasibility. Accordingly, the Commission prepared an actual textbook for high school use on this topic and tried it out in a number of classrooms, with very gratifying results.

This text is of historic importance. It was prepared by a group of authors consisting of both research mathematicians and classroom teachers, and therefore demonstrated that such a group not only could agree on recommendations for the high school curriculum, but could also work together successfully on the preparation of texts for the high school. The enthusiastic reception which this text received, for example, in summer institutes, about which I shall speak later, demonstrated that a recommendation accompanied by text materials which present the recommendation in complete detail, had a much better chance of being accepted quickly and widely than would recommendations alone.

These two facts, that classroom teachers and research mathematicians could work effectively together in the preparation of texts, and that the existence of appropriate texts could drastically shorten the time span between the making of recommendations and their implementation, strongly influenced the work of the School Mathematics Study Group, about which I shall speak next.

The School Mathematics Study Group came into existence in the spring of 1958 as a result of the decision by a group of eminent research mathematicians that the time had come for a massive effort toward the improvement of school mathematics in the United States, an effort in which they should play a substantial but not exclusive role. The group had the blessings of the three principal mathematical organizations: the American Mathematical Society, representing mathematical research; the Mathematical Association of America, representing college mathematics teaching; and the National Council of Teachers of Mathematics, representing school mathematics. An Advisory Committee was organized which represented all parts of the mathematics community: research mathematicians, classroom teachers, experts in education, consumers of mathematics, and so on. Geographically the committee was also representative of the whole country.

A number of different projects were started at once. Each project, after approval by the Advisory Committee, was placed under the supervision and direction of a Panel consisting of representatives of the Advisory Committee as well as other individuals with a special interest in the project in question.

The fundamental objectives of the School Mathematics Study Group, the improvement of the teaching of mathematics in the schools of the United States, were divided into three more specific objectives. The first of these was the provision for an improved curriculum for the schools, a curriculum which would, on the one hand, preserve the important skills and mathematical techniques which experience has shown to be important and useful, and, at the same time, provide students with a deeper understanding of the mathematics underlying these skills and techniques, since it would be this understanding of mathematics which the future will certainly require.

The second objective was to provide materials and direction for the preparation of teachers to enable them to teach such an improved curriculum. The great majority of the school mathematics teachers in this country had never had the opportunity, in the course of their training, to study mathematics from the point of view that would be necessary in any curriculum which looks forward to the role mathematics will play in society in the future. It was agreed that while the work of providing teachers with the assistance they would need to teach such a curriculum would be undertaken by others, in particular through the program of institutes sponsored by the National Science Foundation, to be discussed later, nevertheless the Study Group could well devote some of its efforts to providing materials to be used in this program.

The third objective was that of making mathematics more interesting in order to attract more students to the subject and hold them for a longer time. The needs of the United States for individuals with good training in mathematics and for a general understanding of mathematics by all its citizens are so great that all efforts possible must be made to satisfy these needs. Since it is impossible to force students to study mathematics, every effort must be made to entice them into this field. Very extensive financial support has been provided the School Mathematics Study Group by the United States Government through the National Science Foundation. Education has normally been a function of individual states, and it is a substantial departure from normal procedure for the federal government to be concerned with the school curriculum. This fact, together with the amount of support which has been provided, can be taken as an indication of the great concern the federal government has for the improvement of the school mathematics curriculum.

Most of the money provided through the National Science Foundation has been used to pay for the time of the large number of individuals who have worked for the School Mathematics Study Group.

A major part of the work of the Study Group to date has been devoted to the preparation of sample textbooks. As was mentioned above, it was agreed from the beginning that to make any rapid changes in the schools, it would be necessary to prepare actual texts. At the same time, it was realized from the very beginning that any texts produced should be considered not as a definition of the correct curriculum but, rather, as a concrete sample of the kind of curriculum appropriate for the United States at the present time. Indeed, it was recognized that a wide variety of textbooks would be needed to take care of the wide variety of student interest, teacher ability, and other factors.

It was expected that the major purpose to be served by such texts would be the one mentioned above, that is, to provide a concrete illustration of the kind of curriculum desired, to provide materials which would be useful in the training or retraining of teachers, to provide a source of ideas and suggestions for individual authors of the wide variety of desired texts, and, finally, to provide stopgap materials until a suitable variety could be produced through normal commercial channels. Another aspect of these texts needs to be mentioned. It has been remarked that most of the teachers now teaching in the United States were never given the opportunity in their training program to see mathematics from this new point of view. It was recognized, therefore, that teachers would need some assistance from subject matter experts to be able to teach successfully and comfortably from such a text. It was agreed that, for the time being, it would be best not to make such major changes that a great deal of time would have to be invested by teachers in learning this new way of looking at mathematics. Specifically, the aim was to prepare texts which would require of the teacher no more than what could be learned in a single six-week summer course or a course meeting once a week during the academic year the first time the teacher taught with that text.

With all these considerations in mind, texts were written and are now available for our grades 7 through 12 in revised form, and for grades 4 through 6 in preliminary form. They are being used this present academic year by more than 300,000 students and they seem to be fulfilling satisfactorily the various purposes for which they were prepared. They are providing for a large number of teachers and educators and mathematicians a concrete illustration of what is meant by an increased emphasis on concepts and structures in the school mathematics curriculum.

These texts are being used in connection with many teacher-training programs, both pre-service and in-service. A variety of texts to be published through the normal channels are being prepared by many individuals, although not quite as rapidly as we would have liked. Most of these incorporate ideas and developments exemplified in the S.M.S.G. texts. A large number of schools are using the texts in their regular mathematics program, pending the preparation of similar texts by commercial publishers.

It was mentioned above that a wide variety of texts will be needed in the United States, along the same general lines, it is hoped, as the ones already prepared by the Study Group. It should be explicitly pointed out that these texts do not cover the wide variety needed. To be specific, the texts for grades 9 through 12 were prepared with the college-capable student in mind, that is, the top third of the entire high school population. The texts for grades 7 and 8 were prepared with a larger spread of ability in mind, but even they are not suitable for all students in these grades. We are trying out revisions of our materials for grades 7 through 9 with the slower, or less able, student in mind, but we do not know, as yet, how successful this experiment will prove.

It is our hope that a substantial number of these students will be able to learn good mathematics, though perhaps at a slower pace. If this turns out to be the case, then we shall be able to design courses in shop mathematics, business mathematics, and other similar subjects, for students not going on to universities, which will be far more useful than the courses we now have.

The converse problem, that of the truly gifted student, also faces us. Such students do not find the S.M.S.G. texts sufficiently challenging, but we do not know yet what topics and what kind of materials are best suited to these students.

It was pointed out above that the kind of mathematics or, rather, the point of view toward mathematics taken in these texts is unfamiliar to most teachers. It was agreed, therefore, that considerable effort must be made to provide materials designed to help teachers in this respect. For this reason, each of the texts is accompanied by a commentary which not only contains answers to problems, hints on teaching, etc., but also devotes a good deal of space to a discussion of the mathematics itself and tries to provide the teacher with a suitable background in mathematics contained in the student text. In addition, a series of books explicitly for teachers has been prepared. Some of these are designed to provide general background mathematics and others are aimed at specific S.M.S.G. texts. You may be interested in some of the procedures which we used in the construction of these textbooks and some of the generalizations which I have drawn as to proper methods of carrying on such work.

In the preparation of these texts we took as a basic principle that this should be a joint effort on the part of the groups most closely concerned, classroom teachers and research mathematicians. Each of our writing teams contained equal representation from both these segments of the mathematical profession, and every effort was made to convince each participant that he was on an equal footing with each of the others, and that each individual had special knowledge and competencies which were important for the group effort. At the same time we tried to make it clear that no one individual or no one group possessed special informations or competencies so important as to entitle that individual or group to a veto power over group decisions. When there were disagreements as to topics to be included, or as to methods of presentation, the group was strongly urged to continue discussion until agreement was reached, which almost invariably did happen. Of course, we did include in our writing teams a few representatives of school administration and also a few representatives of the consumers of mathematics, i.e., science and industry. Probably the number of the latter should have been larger, but I see no reason to doubt the basic policy that the bulk of the responsibility should be shared on an equal footing by the classroom teachers and the university mathematicians.

For the most part, the individuals selected to work on our texts were chosen because of their known ability in mathematics and mathematics education, and their known ability to work successfully in a group. In a few cases, individuals were selected not because of their own abilities, but because of the importance of the positions which they occupied. Almost invariably this turned out to be a grave mistake.

A word should also be said about the size of individual writing groups which prepared these texts. The average number of members of any one writing team dealing with a single text was about fifteen. This may seem to be a rather large number to write a single text.

However, it should be pointed out that this group not only prepared a text but also a commentary for the teachers, which involved just as much work if not more. In addition, certain obvious divisions of labor were employed. While some worked on the text material itself, others worked on the construction of problem sets, etc.

To offset the unwieldiness of a group of this size, there are certain advantages in having so many individuals involved in a single text. For one thing, it is much more meaningful to have fifteen individuals agree on a particular matter than it is to have one or two or three reach agreement. And it is much more likely that an agreement reached by fifteen classroom teachers and university mathematicians will find general acceptance. Another advantage of having a large group concerned with a single text is that it supplies a larger pool from which fresh ideas can be drawn, and reduces substantially the probability that good ideas will be overlooked.

Still another very important advantage of having a large group concerned is that it decreases the probability that superficially attractive but fundamentally useless ideas will be included. Almost anyone who might engage in this kind of curriculum work has some pet ideas on certain aspects of the curriculum. Some of these ideas are good and useful, but in general they cannot stand up to a careful and prolonged scrutiny. It is much less likely that an unsound idea will be accepted by a large group than by a small one. It must be emphasized, however, that these advantages of a large group working on a single text are only realized when the procedure mentioned above is followed, with disagreement handled by further and more careful discussion and study until agreement is reached.

Let us return to the problem, mentioned earlier, of the retraining of teachers to prepare them to teach the improved curriculum. This has proved to be a serious problem, not only for the School Mathematics Study Group but for all the curriculum improvement groups working in the United States. Most of our teacher-preparation programs have failed to provide the kind of training necessary to handle an improved curriculum successfully.

The S.M.S.G. procedure in its initial tryout period was to bring together small groups of teachers, all teaching the same course, about once a week for a session lasting about an hour. At this session a subject matter expert, usually from a nearby college or university, discussed with the teachers the mathematics that they were at that time teaching to their students. Questions of pedagogy were left to the teachers, and the discussions were confined to the mathematics itself and to providing for the teachers a deeper understanding of the mathematics and its place in the curriculum. This procedure turned out to be highly successful. Teachers who taught an S.M.S.G. course with this kind of subject matter assistance were able to teach the same course the next year without any further assistance and, at the same time, with better results.

The procedure of providing in-service subject matter assistance to teachers while they are teaching a new course for the first time, is being widely used by individual schools that are now adopting new curricula. In some cases the subject matter assistance is provided by university mathematicians, but in many cases, by teachers who have already been through the process. In particular, we are discovering that it is quite effective to have the in-service subject matter assistance for junior high school teachers provided by experienced high school teachers, and, similarly, to use experienced junior high school teachers to provide subject matter assistance for elementary school teachers.

There is, however, a very large program financed by the federal government, through the National Science Foundation, which provides subject matter training for teachers. The program antecedes and is independent of the curriculum improvement projects. The assistance provided by the National Science Foundation is in the form of financial support for programs called institutes, which are planned and operated by individual colleges and universities. Any university is free to submit proposals for institutes and these proposals are evaluated for the Foundation by panels of educators drawn from all parts of the country.

One type of institute is a summer program usually lasting for six weeks, although some exceptions to this length can be found, with from twenty to about fifty high school teachers participating. The participants are given a stipend as well as travel and living expenses. A typical institute consists of two or three intensive courses in mathematics, usually chosen for their relevance to the high school curriculum but not necessarily tied closely to any particular new curriculum. The objective is to increase the mathematical competency of the high school teacher. In some cases, the program may include seminars or workshops devoted to the new curricula, but the major emphasis is on the courses in mathematics.

In recent years there have been, each summer, between three and four hundred such institutes for high school teachers of science and mathematics, with approximately one hundred of them devoted exclusively to mathematics.

In the past few years the National Science Foundation has also supported a small number of similar institutes for teachers and supervisors of elementary school mathematics. The number of such institutes will undoubtedly increase in the near future.

In another program, also financed by the National Science Foundation, in-service institutes are operated by a variety of universities. In such an institute, the participants are all from the immediate geographical locality and they meet once or twice a week, in the evening or on Saturday morning. As in the case of the summer institutes, the program for an in-service institute is devoted primarily to courses in mathematics designed, again, to increase the teachers' mathematical competency. For these in-service institutes, the participants receive travel expenses but no stipends.

In still another type of institute, the participants attend a university for a full academic year, and sometimes for the preceding or following summer, or perhaps both. Again the program is devoted primarily to mathematics. Participants in these institutes, of which there are about a dozen devoted exclusively to mathematics this year, receive a stipend comparable to, but rarely equal to, their normal salary, and also receive travel and living expenses.

In most of these institutes, the courses provided have been especially designed for the participants, since the usual university courses are rarely appropriate for teachers who have been away from their studies for a number of years. It has been the hope of the National Science Foundation that such courses would be incorporated into the regular summer offerings of our universities, and this seems to have taken place to some extent.

For teachers who have already attended some institutes, or whose mathematical training is above average, the National Science Foundation has another program of individual summer fellowships which allow a high school teacher to attend a university for one, two, or three summers, pursuing his own individual program.

Despite the enormous effort and expense devoted to the upgrading of the present teachers in the United States, we seem to be losing ground. Each year we turn out of our teacher-training institutions a large number of teachers who are not prepared to teach the new mathematics curricula. As fast as we retrain our current teachers, we are faced with an equal number of fresh graduates who need the same retraining.

In addition, we are faced with a problem of a larger order of magnitude when we consider the mathematics curriculum for the elementary schools. Here the number of teachers are so vast that there is no hope of upgrading their training through any program of summer or in-service institutes.

In the last few years, the college mathematics teachers not only have faced up to these two problems but have actually started to take some steps to meet them. The major efforts along these lines are being carried out by the Committee on the Undergraduate Program of the Mathematical Association of America. This Committee is considering all aspects of the college mathematics curriculum and one of its subcommittees has the special task of considering the mathematics preparation of future teachers, at the elementary and the secondary level, and also at the college level. The Committee has recommended mathematics courses most appropriate for mathematics teachers at these various levels and is now engaged in stimulating mathematicians to prepare textbooks suitable for such courses and to have them tried out in experimental teacher-preparation programs. The Committee believes that the mathematics preparation required of teachers can be specified rather exactly. In fact, it has recommended that, contrary to current practice in this country, the preparation program of teachers be tied very closely to the courses which the recipients of this training will actually teach in our schools. More specifically, the Committee recommends that the basic minimum preparation for any prospective school mathematics teacher should be such that the teacher will eventually teach and, in addition, will have a thorough understanding of the basic concepts of mathematics that his students will study at the next educational level. This recomendation thus takes into account an important characteristic of all proposed new curricula in mathematics. They all stress greater emphasis on basic concepts and, since these concepts persist throughout the entire curriculum, it is important for a teacher to understand how they are treated in courses which the students will take later, so that their presentation at any particular level will be in harmony with later presentations of the same concept.

It is likely that these recommendations of the Committee on the Undergraduate Program will have considerable influence on the preparation of mathematics teachers. On the one hand, these recommendations are made by a representative, influential group. On the other hand, the various state agencies which issue certificates entitling individuals to teach in the public schools of the United States have issued recommendations for the training of mathematics teachers which are in the same spirit and direction as the recommendations of the Committee on the Undergraduate Program. These two forces acting in harmony will undoubtedly produce a much improved program of preparation for mathematics teachers within the next few years and provide us with a much augmented pool of teachers able to handle the new curricula.

To summarize, we in the United States have embarked on a massive effort to improve the school mathematics curriculum, a curriculum which has stagnated for the last two generations. The reason for making this effort is not dissatisfaction with the past but rather the realization that the future will require far more understanding and skill in mathematics on the part of all our citizens.

This improvement effort has many facets, since progress depends on improved text materials, improved training of prospective teachers, and upgrading of our present teachers. Work on all these facets of the problem is proceeding on many fronts and involves all parts of the mathematical profession. Some progress is being made, but much remains to be done and most of the programs which I have described will undoubtedly continue into the indefinite future.

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Discussion Following Address by Professor Begle (U.S.A.)

Professor Begle's address on the special reforms in mathematics teaching in the U.S.A. was greeted with the questions and comments that follow.

Prof. Alfaro (Costa Rica): It seems to me that the recommendations and suggestions given by Professor Begle's group merit an adequate translation since the little that we have managed to find out about the work does not seem to be correctly presented.

As for the pressing need to have teachers reach a certain level of competency, which is undoubtedly necessary if we try to make rapid changes in our teaching, I imagine that very few countries are in a position to do much about this. Moreover, it should be remembered that even when teachers fulfill the requirements of the positions they hold, a great deal depends upon the fact that tenure in their positions is guaranteed by law. This means that for the next thirty years, teachers will not be qualified to teach the new work unless they submit to re-education either by means of summer courses or through journals. Since the first is not always possible, I think that the translations of the recommendations of the College Entrance Examination Board of the U.S.A. would be very helpful—not only the current reports but also the supplements—in order to keep the teacher up to date on developments.

Prof. Begle (U.S.A.): A meeting of the committee on translations has been scheduled for this afternoon. Although I do not have the authority to assign translations to this or that country, you may be assured that any request of this type directed to our group would be honored with good will. In any event, it seems to me that it would be very risky to translate material that is just in the process of being developed, and in an experimental state.

Prof. Robinson (West Indies): From the beginning of the week, I have been somewhat worried about the same problem that Professor Begle has been discussing, that is, the problem of creating an opinion favorable to the new point of view in teaching, and this, it seems to me, will raise certain difficulties. As far as I am concerned, I wish to know what goal is to be achieved in such a change.

Is it that the students will be able to progress more rapidly in the university or will they become better research mathematicians? If any such information were available, we would be in a better position to judge these programs and experiments, with a view to introducing changes in our regions.

Prof. Fehr (U.S.A.): I will attempt to give a partial answer. It has occurred to teachers that thanks to changes that have already come about, mathematics plays an essential role today, and it is not just a matter of method.

Prof. Stone (U.S.A.): I believe that the best evidence of the value of modern methods would be that mathematics had greater power to attract students. For, if this were not the case, then we have not achieved the necessary objective of giving vitality to a study which is needed by the entire technical world, even though it does not seem to me that the necessity of modernizing arises as much from the academic as from the social aspects of the subject. Nor should we forget the economic and social advantages which a better preparation in science provides and which are made possible by the modern methods. On the other hand, I believe that at least twenty years are needed to verify whether or not the changes we are considering today are beneficial.

Prof. Bundgaard (Denmark): In Denmark, we had the following experiences: When we were introducing changes in 1956 and 1957, the purpose and extent of the reforms were widely publicized through the medium of periodicals. In this way we obtained the support of many people. In the four years since we embarked on this new enterprise, it can already be observed that the number of students in this branch of science has increased tremendously, the annual increase being of the order of 20 to 25 per cent. I believe that not only is it possible to interest the authorities, but public opinion can also be orientated, and I hope that before long you will try to do this in your countries.

Prof. Coleman (Canada): In Ontario, we believe it is necessary first to convince the teacher of the advantages of change. Furthermore, in our educational system, no decision is reached without consulting the teachers; and it could happen that the teachers might be in opposition because the changes about which they are being consulted appear to them to be foolish and therefore unnecessary. Consequently, we decided to charge the most experienced and respected teachers with the task of experimenting with the new material. During the past year, 65 teachers conducted the first experiment with the new curriculum, and of this number 60 ended the year condemning the old methods. I believe, therefore, that the final decision about the advantages of the reform of mathematics must lie in the hands of the teachers.

At a previous meeting, Professor Laguardia mentioned that it would be helpful if the units of measure which are used in the country for which the book is destined are employed in any translation of it that is being made from the English. I believe, moreover, that textbooks that are being translated should be introduced to teachers little by little to facilitate their study. It would be advisable, furthermore, to call meetings of the teachers a year after the books have been distributed so that they can compare notes.

Prof. Begle (U.S.A.): The texts, at the moment, are available in English, and the idea of translations of these texts and a review of them seems to me to be a good one.

Prof. Garcia (Puerto Rico): A group of us in Puerto Rico is responsible for the translations of the ninth grade material; this has been distributed among you in mimeographed form. Since, in our country, the English system of measure is in use, it is the one that we employed in the translation. Naturally, the texts would have to submit to some modification to be usable in other countries.

*THE ROLE OF MATHEMATICS IN PHYSICS, FROM THE VIEWPOINT OF SCIENTIFIC EDUCATION**

Professor Laurent Schwartz (France)

For a long time, theoretical physics utilized only relatively limited sections of mathematics, essentially partial differential equations along with some classical methods which amounted to successive approximations or a reduction to integral equations. The present development of theoretical physics, on the other hand, requires an extensive use of mathematical tools. All the methods of analysis must take part, and also algebra, the theory of the representation of groups, and even more recently, the theory of cohomology and analytic functions of several complex variables. This accounts for the fact that the study of the role of mathematics in physics has become a problem of the first rank and makes evident the need for important reforms in this field.

I. Mathematicians of the present day do not know enough physics.

At the time of Poincaré, the majority of mathematicians knew the principal problems of mathematical physics. It was for them a source of ideas and new methods, and physics contributed to the progress of analysis. Today the situation is different and there are many reasons to account for this change. On one hand, the considerable increase in the volume of mathematical and physical knowledge, which tends to encourage greater specialization and which very often separates mathematicians from one another, cannot help being an agent for separating mathematicians from physicists. On the other hand, the fields of mathematics which have received the greatest development in the last twenty years, especially algebra and algebraic topology, do not have any relation to physics. For this reason, the mathematician who is placed in contact with these new fields has to devote to them most of his time and very soon loses all interest in physics, which he will come to regard as an unpleasant task, one that is necessary for satisfying graduation requirements. What I have just said does not seem to be relevant to this conference, but I think it is a very important point. To be able to offer the physicists good mathematics adapted to their re-

*Address translated from the French.

quirements, there must be a sufficiently large group of mathematicians interested in physics. We have in this situation a vicious circle, in that since the physicists do not have a good mathematical background, courses in physics are often of little interest to the mathematician because their complete lack of rigor makes them unacceptable beyond certain limits. The mathematician who sees, in a physics textbook, the symbol Σ in place of the integral, and who knows the difficulties that integration can present, experiences an unavoidable impression of lack of rigor. In many cases, this will be of no great consequence, since he will know, without much trouble, the way to remedy the situation. But if Poisson's formula for Newtonian potential is demonstrated for him in a cavalier manner,

$$\Delta U = - \rho$$

with U representing the potential of charge ρ, then, because of his awareness of the great importance and effect of this formula, it will not be possible for him to experience an attraction to modern physics. In the same way, it is shocking that quantum mechanics should be of interest to so small a number of mathematicians, including those who are specialists in the theory of operators in Hilbert spaces. This is due, in large part, to the fact that the physicist develops quantum mechanics as if spectral decompositions were not more difficult in general than the case of finite-dimensional space. The mathematician is sufficiently aware of these difficulties and is so thoroughly convinced of the important role that they play in theoretical physics that he cannot but be deeply shocked by this complacency; therefore, certain mathematicians who would have made great progress in theoretical physics because of their knowledge, in reality have not had any interest in the subject. This difficulty is in danger of becoming even more pronounced in the future. Theoretical physicists, in order to advance, cannot submit to the rigorous discipline which the mathematician would impose on them, and without which, however, they would have discovered practically nothing. They are forced to plunge with temerity and audacity into domains that have scarcely been explored, without giving much thought to method. Moreover, we cannot criticize their lack of concern. I think, however, I ought to point out the unavoidable risks in this situation, insofar as increasing the interest of mathematicians in physics and their possible future collaboration. This can only happen if those parts of physics which can be so presented are given with desirable mathematical rigor.

It might even be quite desirable to create a special certificate in physics for mathematicians in connection with the mathematical degree of licentiate. This certification would be in charge of a physicist and a mathematician. Such training could be based unhesitatingly on the greatest possible number of mathematical tools. Potentials could be defined by convolutions; Poisson's formula could be proved by using the Laplacian of $1/r$ in the sense of distributions; the various formulae of electromagnetism would be derived from a mathematically correct solution of Maxwell's equations, making use of the elementary solution, in the sense of distributions, of the wave equation. Diffraction, in optics, would systematically use the Fourier transformation. We cannot ignore the difficulties of such an enterprise, since the mathematical tools to be used are not known to the mathematician at the beginning of his career. Nevertheless, a serious attempt should be started in this direction, since, as far as I know, nothing has been done about it in any country.

II. The physicists do not know enough mathematics.

It is, however, the lack of mathematical knowledge on the part of the physicists that is one of the main obstacles in the harmonious development of theoretical physics at the present time. Responsibility for this situation belongs to the method of teaching mathematics which has been in vogue all over the globe up to the last few years. The only regular mathematical training the physicist received, up to the present, was differential and integral calculus, taught to mathematicians and physicists alike. This integral and differential calculus might have been excellent, perhaps, for the development of mathematical ability, but it is absolutely of no use to the physicists. There was an emphasis on purely theoretical concepts which could be used by the mathematician, since he would have time later to complete his training; but these were useless to a physicist. On the other hand, the physicist was not given the principal mathematical tools which he would need in all his future work. He was therefore forced to prepare himself in mathematics on the side, and as a consequence, he was far removed from the development of modern mathematics. He had heard about the theorem of Weierstrass-Bolzano, but he did not know anything about the Fourier integrals or convolutions. He had learned all the subtleties of finding derivatives under the integral sign, but knew nothing about Hilbert space or the theory of groups. The mathematician, of course, was in the same situation, but he was better prepared to round out his training by getting the certificate of higher analysis or by reading modern books. In reality, it must be admitted that the physicists working alone have filled the gap in this situation in a remarkable way, since a number of them have managed to acquire as much knowledge as the mathematicians in the subjects in which they were interested. In general, though, they lack certain fundamental techniques. The most serious, without a doubt, at this time, and the lack which represents a wider cleavage between mathematicians and physicists, pertains to linear algebra. In reality, it is also new to the mathematician. The generation which preceded ours had a good knowledge of Banach spaces and of Hilbert spaces, but the present way of handling vector spaces of finite dimension with their duals and their tensor products was practically unknown. Today any mathematician who has recently been trained handles linear and multilinear algebra quite easily, while the majority of the physicists ignore the ABC's of the subject. In classical mechanics or theoretical physics, they have to handle covectors, matrices, tensors, but they always do so using the famous T_k^{ij} with raised and lowered indices, which has provided and still provides an excellent form of fast calculation, but nevertheless masks a profound truth.

A tensor of any type is, first of all, an element of certain tensor space and the G_k^{ij} are nothing but their coordinates with reference to a base. Experience tells us that the physicist practically never knows if he is dealing with a vector space or its dual, if he has to do with a bi-linear form represented by a tensor doubly covariant, or with a linear mapping represented by a matrix or a mixed tensor. The appearance of a quadratic euclidean or lorentzian form seriously complicates the situation, since it defines an isomorphism between the space and its dual which confuses the situation unless an effort is made to keep solid bases. Let me give a simple example.

Let E be a vector space with a Lebesgue measure dx; the Fourier integral is defined by the formula

$$(1)\qquad g(p) = \int_E f(x) \exp(-2i\pi \langle x,p \rangle)\, dx$$

where f is a function on E and its Fourier image g is a function on its dual E'. The expression $\langle x,p \rangle$ designates the scalar product between $x \in E$ and $p \in E'$. This Fourier transformation does not need any other hypothesis, and in particular does not introduce any quadratic form. If dual bases on E and E' respectively are so selected that the coordinates of x and of p are respectively $x_1, x_2, \dots, x_n$, and $p_1, p_2, \dots, p_n$, the formula would still be written as

$$(2)\qquad g(p_1, p_2, \dots, p_n) = \iint\dots\int f(x_1, x_2, \dots, x_n) \exp[-2i\pi(x_1p_1 + x_2p_2 + \dots + x_np_n)]\, dx_1, dx_2, \dots, dx_n$$

If, on the contrary, we had taken a quadratic form on E (for example, of signature (3,1) where E is the four dimensional space corresponding to restricted relativity) then it would be possible to define a Fourier transformation of the following type:

$$(3) \qquad h(y) = \int_E f(x) \exp [-2i\pi (x,y)] dx$$

The function f and the function h are both defined on the same space E ; the vectors x and y are on E, and $(x|y)$ shows the scalar product defined by the quadratic form.

In the lorentzian case just mentioned, if we were to take a lorentzian base and if we were to designate the coordinates of x by x_1, x_2, x_3, x_0, and those of y by y_1, y_2, y_3, y_0, the formula would be written as follows:

$$(4) \qquad h(y_1, y_2, y_3, y_4) = \iiiint f(x_1, x_2, x_3, x_0) \exp [-2i\pi (x_1 y_1 + x_2 y_2 + x_3 y_3 - x_0 y_0)] dx_1 \, dx_2 \, dx_3 \, dx_0 .$$

We pass from formula (1) to formula (3) quite naturally because of the correspondence between the space E and the space E', defined by the quadratic form.

Even though this Fourier transformation is in common use in theoretical physics, I believe there are few physicists who have really understood the simple algebraic structure, just described, which is completely hidden by the handling of indices. Many physicists, for example, use the transformations (3) and (4), in the belief that f is defined on the space of "configuration" E and h on the space of "phase" E'. The attendant confusion is the more regrettable because the formula written with indices would not be invariant. Examples of this kind can be multiplied by the hundreds.

The absence of linear algebra makes it very difficult for the mathematician to read any book on modern physics, and even though he does not know it, it obscures considerably the thinking of the physicist. All sorts of adjoint tensors of complex conjugates used in theoretical physics provide other examples. In the usual topological vector spaces, the role played by the transpose of a mapping is known, but one knows that when it concerns Hilbert spaces the adjoint of the mapping is also introduced and the transformation $A \to A^*$ is then antilinear. The direct algebraic definitions of the transpose and of the adjoint are trivial. The physicist defines them almost always with bases using indices, which complicates the situation uniquely. But it is more ominous if they are not matrices, but tensors of some sort. Let E be a vector space over the complex field; we say that an anti-space of E is one for which there is another vector space E^* over the complex field, and an anti-isomorphism "*" between E and E'. There are an infinite number of models of anti-spaces, but any two anti-spaces are always canonically isomorphic. We could always, for example, take for E^* the set E itself. The mapping "*" would then be the identity mapping, with the vector structure of E^* defined, for addition, by the same law as for E, and for multiplication by the law

$$(ax)_{E^*} = (\overline{a}x)_E$$

There are, however, other methods, and these are quite different. For example: If E is the space L^2 of square functions which are integrable over the real line R, we would take $E = E^*$, with the same law of addition and multiplication, and the anti-isomorphism "*" being defined by the formula

$$f^*(x) = \overline{f(x)}\ ; \quad x \in R$$

If one takes on E a base e_i and on E^* the corresponding base e_i^*, the correspondence between the coordinates of two elements x and x^* is given by

$$x = \sum_i a_i e_i\ ; \quad x^* = \sum_i \overline{a}_i e_i^*\ ; \quad a_i \to \overline{a}_i$$

This operation is constantly made in theoretical physics but in the following erroneous way. It is simply said: Let us consider the tensor we get by replacing all the coordinates T_k^{ij} of the initial tensor by the complex conjugates $\overline{T}_k^{ij}$. This conceals the fact that we have passed from a vector belonging to a vector space E to a vector of another vector space, the anti-space E^*. In a general way, almost all the linear transformations of physics are defined by means of matrices starting with a base, or, except in rare cases, by a linear transformation which is defined by matrices and is of a simple invariant type. The problem is more easily solved by vector relations in the vector space itself, independently of a base. Moreover, the use of bases can be of some effectiveness in spaces of finite dimension, but leads to much greater difficulties in Hilbert spaces of infinite dimension. However, a direct definition of the linear transformations used can be quite convenient.

This gap in connection with linear algebra will create, for many years to come, great difficulties in the understanding of mathematicians and physicists for each other's work, even though these difficulties are purely artificial and on the surface, and do not relate to any serious obstacle in theoretical physics. But it is encouraging to note that this gap is rapidly being filled in; in reality, in France, linear algebra is systematically being studied from the time of entrance into the Faculty, in the preparatory class (i.e., general mathematics), and all the young people who are directed toward mathematics or physics handle linear algebra with increasing ease, and are very well satisfied with their ability to do so. The generation to come will not be aware of this purely artificial problem. It is, however, important to favor, in all possible ways, all books, written by physicists and for physicists, which popularize linear algebra.

On the other hand, there is a difficulty of the same nature but one which will last much longer, and this is the one of handling differential forms and Stokes' formula. The experts in mechanics, the physicists, and the engineers have a very good knowledge of the theory of vector fields, gradient, divergence, the rotor, and all their mutual relations, but they are practically completely ignorant of differential forms and the integration of these forms. Now methods of this type have become so much more useful than vector fields for the mathematician that practically all mathematicians today ignore vector fields. The breach over this point has become even wider than it formerly was. It should be noted that differential forms and Stokes' formula are incomparably more complicated than linear algebra, and cannot be presented to physicists at the beginning of their training. They cannot even be presented later on, except quite imperfectly. In order to be presented effectively, they should be preceded by an introduction to differentiable varieties, which are at this time a concept that is still relatively too abstract and, in all its generality, of doubtful interest for the physicist. We should find a means of resolving this situation, for here too, if the theoretical physicist makes less use of antisymmetric tensors and more use of differential forms in connection with restricted and general relativity, the situation would be clarified more effectively.

III. The present training of physicists in France and proposals for improving it.

The requirements for a degree of licentiate in France was changed radically in 1958. As far as the three degrees in mathematics, applied mathematics, and physics are concerned, the idea of the reform is as follows:

A licentiate in mathematics today must know many more subjects than formerly; the old certificate of differential and integral calculus is no longer sufficient. If it is true that he will need less of the theory of curves and surfaces than formerly, if it is true that he will always have to know uniform convergence and all that which goes with it and analytic functions of a complex variable, he will also be required to know linear algebra perfectly and some multilinear algebra, general topology, something about groups, rings, and fields, and some theory about differentials in the modern form. This is the reason for bringing up to date the old program of differential and integral calculus, and at the moment it is the subject of two different licentiate certificates, Mathematics I and Mathematics II. In the attached schedule are included the programs for these certificates. Lebesgue integration is not always a part of these programs, but certain professors treat it, in part, in their courses in Mathematics I.

The totality of the subjects of these two certificates necessary for the licentiate is far from being sufficient for the researcher in mathematics; but he will be able to complete his studies by taking, outside of program, a certificate in Mathematics III, the content of which is not determined by a Ministerial resolution but satisfies the requirements set up by professors in the various faculties and which, in effect, presents Lebesgue integration and the measure theory of Rado, distributions, convolutions, Fourier transforms, and a study of somewhat greater depth of differential equations, essential concepts of Banach and Hilbert spaces, differential varieties, Stokes' formula, topics on groups, and eventually Lie groups and Lie algebras. He can also take the courses in higher mathematics in which the program is selected ad libitum and depends on the years when the courses are offered. On the other hand, it is essential that the mathematician have a minimum knowledge of physics. It is not possible to demand from the licentiate in mathematics that he obtain the old certificate of general physics, since it has become a very heavy one because of progressive modernization. This certificate is at the moment divided into three different certificates which are, in essence, thermodynamics, optics, and electricity.

We demand from the mathematician today only one of these certificates, that is, knowledge of one of the three main branches of physics, and that of his own choosing. We think it is impossible to do otherwise, at least for the greater number of licentiates. Nevertheless, certain inconveniences of the present situation are evident. This program may provide good mathematicians, but they will have a very limited knowledge of physics. It would be necessary, as I have mentioned in section I, to replace one of the three elective certificates in physics by a certificate in physics which is planned especially for mathematicians and prepared, perhaps, by mathematicians and physicists working in collaboration, and in which modern and rigorous mathematical methods are used. This matter involves an idea not yet put into practice, one which would not solve the whole of the problem because the training in physics of the mathematician should not be solely at the level of the licentiate but should be at a higher level, perhaps postgraduate.

Let us now examine the training of the physicist.

Previously, the certificate in general physics formed the basis of the training offered for the physicist, but it suffered the same evolution that differential and integral calculus has had; it became wholly inadequate for the physicist. For this reason it has been divided into two-year periods and split into three certificates:

thermodynamics, and physical mechanics; optics; electricity. These three certificates are required for the licentiate in physics, just as the two certificates, Mathematics I and Mathematics II, are required for the mathematician. The physicist must also obtain a certificate in either chemistry, electrotechniques, or electronics, and at least one certificate in mathematics. The latter bears no resemblance to the old certificate in differential and integral calculus and is called: Mathematical Techniques for Physics (M.T.P.). This is an elementary certificate provided especially for physicists. The courses are generally given by mathematicians but sometimes by physicists, and emphasize the tools which are useful to physicists and engineers (see the appendix of this paper for details). In the outline of the program, one will note that the following topics are requirements: the development of the equations of Laplace, the equation of vibrant chords, some theorems on problems in limits, elements of the calculus of probabilities. These are not required for the licentiate in mathematics. The mathematicians will not study the calculus of probabilities of a type which is somewhat advanced and rigorous unless they are working for specialized certificates at a postgraduate level, whereas the physicists need the concepts of probabilities from the beginning of their studies. The program for the course in "Mathematical Techniques of Physics," according to official descriptions of it, enjoys a certain amount of flexibility and therefore there are many variations of it throughout France. Vector analysis generally includes concepts of differential forms and Stokes' formula; supplementary topics on integration include certain notions of Lebesgue integrals and a precise statement of the Lebesgue convergence theorem. The Fourier series are treated in the manner of the series development for orthogonal functions in a Hilbert space. A course in "Mathematical Techniques of Physics" has no relation at all to a course on mathematics for mathematicians. The future physicist or engineer is given a large number of useful concepts and an extensive body of mathematical knowledge, but theorems are presented without demonstration. While evidently preserving for mathematics whatever is fundamental and also a large enough number of demonstrations so that the students will not lose track of the need for the process, they will be presented with a large number of results that they will accept (with precisely stated references), and they will be authorized to make use of theorems which they are unable to prove. This method of teaching a profusion of topics will serve to fix the essential structures of the mathematics which they are studying. (It should be noted that linear algebra is not part of the program of the course "Mathematical Techniques of Physics" because it is included in that for "General Mathematics," i.e., in the preparatory section for the licentiate, and corresponds to the first year in the

Faculty, and is common to the programs of mathematicians and physicists alike.)

If one could grant that the subjects in the certificate for "Mathematical Techniques of Physics" are sufficient for the experimental physicist, they are completely insufficient for the theoretical physicist; but we note the existence of options in both of the certificate programs in mathematics and physics. Because of these options, each student can look for those subjects which best suit his talents. There is, however, an option which is strongly recommended for the physicist, and this is the certificate in "Mathematical Methods of Physics" (M.M.P.), which I am assigned to teach in the Faculty of Paris. (I shall not try to give the subtle distinction between "Techniques" and "Methods"; but shall simply say that the M.M.P. corresponds to a higher level of mathematics intended for physicists and engineers.) The program of study for this certificate is given in the appendix to this paper. Actually it is officially considered to be too extensive, and the instructor presents a part of the program, of his own selection. Integration, if given in full, includes Lebesgue integration, but with all demonstrations of difficult theorems omitted. The theory of distributions is treated with as little topology as possible. Convolutions are developed in connection with Fourier and Laplace integrals, with the Fourier integral being considered from the point of view of tempered distributions. In connection with Hilbert spaces, we can give the theorem of projection, Hilbert bases, and the whole of the theory of orthogonality. If the instructor decides to go into details on this question, he can go deeply enough to give the spectral decomposition of Hermitian operators or normals, and to make mention of unbounded operators. The special functions are essentially those of Bessel and Legendre, in addition, naturally, to Euler functions. Tensor calculus might include the elements of tensor analysis and also some work on differential forms. Naturally, in anything involving partial differential equations of the second order, it would be a matter of showing the role which problems using limits play in physics and of giving some indication of the way in which Hilbert methods can be used to handle them along with Green's kernels and the series development of functions. I personally have presented, in these last few years, integration, distributions, convolutions, the Fourier integral, and special functions; or, integration, distributions, convolutions, special functions, and partial differential equations of the second order.

The certificate "Mathematical Methods of Physics" corresponds to three hours of instruction a week during the first term, and, as a consequence, the program cannot be made too heavy. It is permitted, of course, and even recommended (and this is what I have done in all previous years) to continue the instruction in this material in the second semester, in a course without examination for a certificate, but treating several parts of the program which could not be studied during the legal period for the preparation for the certificate.

There is, moreover, another possible certificate called "Mathematical Methods of Physics II," the other course then being given the title "Mathematical Methods of Physics I." This certificate (M.M.P. II) does not count toward the licentiate certificate for teaching, but is intended only for researchers and for the doctorate. The program is of postgraduate level, and is rather arbitrary in its selection of topics. It is meant to give the future researcher of physics as solid a mathematical training as possible.

It should be noted that the physicist can elect, for the certificate in physics, one of the certificates of the licentiate in mathematics, such as Mathematics I or Mathematics II, instead of the M.M.P. This often presents a very interesting combination. In effect, the certificate M.M.P., like the T.M.P., describes a great many results and gives a precise statement of structures, but, on many occasions, it omits the most subtle demonstrations. The student who elects to take a certificate in Mathematics I or Mathematics II will obtain from his mathematical education the notion of a complete development of all the demonstrations, something that will prove useful later on. He will acquire fewer results that can be applied to his physics, but he will know how to prove all of them.

It is desirable that a number of physicists choose Mathematics I or Mathematics II instead of the M.M.P. or even that they take both. It should be emphasized that the certificate of M.M.P. can be chosen by the mathematician for the licentiate in mathematics and that the students of the École Normale Supérieure, who are at our highest level of study for the licentiate, all have been taking, during the last few years, the three certificates, Mathematics I, Mathematics II, and the M.M.P., and therefore are not required to take the T.M.P., since they are destined to be mathematicians or physicists. This practice is an excellent one which in the coming years will produce a generation of mathematicians and physicists who will have been given the same fundamental and solid training.

The course of study for the licentiate in "Applied Mathematics," which is somewhere between mathematics and physics, and which trains mechanics and engineers, is given in the appendix to this paper.

The present system has been in effect since 1958 and seems to be having good results, but it will probably be altered regularly. Actually, we, in France, would like to have a permanent commission on "Reform of Teaching in the Higher Schools" which would study regularly all changes that might benefit our system of education.

Courses of Study for the Licentiate in France

I. Certificate in Mathematics I

1. Sets; groups; linear algebra; elements of higher algebra; quadratic forms.
2. Concepts of topology: compact spaces; continuous functions in a compact space; metric spaces; examples of functional spaces; method of successive approximations; applications to a system of differential equations; series and infinite products.
3. Riemann integration of numerical functions in R^n; change of variables in multiple integration; continuity; integration; derivation of series and of integrals dependent upon a parameter.
4. Prehilbert spaces; Fourier series.

II. Certificate in Mathematics II

1. Differentials; implicit functions; Stokes' formula in two or three variables; primitive of a differential form of first degree. (Systems of ordinary differential equations; first integrals; elements of the calculus of variations.)
2. Analytic functions of a complex variable.
3. Differential geometry of curves and surfaces in R^3; Riemann spaces of two dimensions.
4. Foundations of the theory of integers, rationals, reals, complex numbers; the field of integers modulo p.
5. Euclidean geometry; affine geometry; projective geometry; foundations of elementary geometry; examples of non-euclidean geometry.

III. Certificate in General Mechanics

1. Topics in kinematics and kinetics; forces; fundamental principle and general theorems of Newtonian mechanics; work and virtual potential; theorems on energy; problems in statics and in the dynamics of solids.
2. Currents; friction; gliding, with equations related to concrete problems.
3. Equations of Lagrange and canonical equations; introduction to the mechanics of continuously deformable media.

IV. Certificate in Technical Mathematics

1. Vector analysis.
2. Topics in integration.
3. Linear differential equations.
4. Analytic functions.
5. Fourier series.
6. The Laplace equations, $\Delta V = O$, in 2 and 3 dimensions.
7. The equation of vibrating chords.
8. Elements of the calculus of probability.

V. Certificate in Mathematical Methods of Physics I

The program is to be planned by each Faculty authorized by Ministerial decree to handle this certificate, subject to review by the Council on Higher Education. The following list represents about one half the usual program:

Integration; distributions; convolutions; Fourier and Laplace integrals; Hilbert space; special functions; elements of tensor calculus; restricted relativity; the Lorentz group; partial differential equations of the second order.

Courses of Studies for the Various Programs

I. Mathematical Sciences

1. Mathematics I.
2. Electricity or thermodynamics and physical mechanics or optics.
3. Mathematics II.
4. General mechanics.
5. A certificate of higher studies oriented toward either mechanics, probability, or astronomy, or numerical calculus, or algebra, and chosen from a list set by each Faculty of Science by Resolution of the Ministry of Education, and subject to review by the Council on Higher Education.

II. Applied Mathematical Sciences

1. Mathematics I.
2. Mathematical techniques of physics.
3. General mechanics.
4. Electricity or optics, or thermodynamics and physical mechanics.

5. and 6.
Two certificates of higher studies orientated towards mechanics or probability or astronomy or numerical calculus or algebra and chosen from a list set for each Faculty of Science by Resolution of the Ministry of Education, and subject to review by the Council on Higher Education.

III. Physical Sciences (Physics I)

1. Mathematical techniques of physics.
2. Electricity.
3. Optics.
4. Thermodynamics and physical mechanics.
5. Mineral chemistry, or organic chemistry, or systematic chemistry.
6. General mechanics or astronomy, or mathematical methods of Physics I, or physical crystallography, or any other certificate of higher studies selected from a list set for each Faculty of Science by Resolution of the Ministry of Education, and subject to review by the Council on Higher Education.

IV. Physical Sciences (Physics II)

1. Mathematical techniques of physics.
2. Electricity.
3. Optics.
4. Thermodynamics and physical mechanics.
5. Electrotechniques or electronics.
6. General mechanics and astronomy, or mathematical methods of Physics I, or physical crystallography, or any other certificate of higher studies selected from a list set for each Faculty of Science by Resolution of the Ministry of Education, and subject to review by the Council on Higher Education.

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Discussion Following Address by Professor Schwartz (France)

Professor Schwartz' address stimulated a lively discussion, with the main points at issue being those indicated in the following remarks.

Prof. Dominguez (Argentina): I would like to offer some observations on the theme developed so brilliantly by Professor Schwartz. The first concerns the importance of linear algebra, the study of which seems to me to be of capital utility. I also wish, however, to emphasize the enormous importance of the study of functions of many complex variables, a study which is somewhat neglected in our schools.

My third observation is that I beg to differ with Professor Schwartz. He has said, and said with strong conviction, that there is no book on physics in which the Fourier transformation is explained with sufficient rigor; he has assured us that there is no book in existence which explains this topic with sufficient clarity. This is absolutely false. There is such a book—a remarkable book—entitled Mathematics and Quantum Mechanics, which explains this and other abstruse topics. Its author is Professor Schwartz!

Prof. Coleman (Canada): I have read some texts used in France and I believe that the teaching done by the physicist in these books is beset with many problems similar to those of secondary school teaching. In Toronto, the mathematical preparation of our physicists is quite similar to that described by Professor Schwartz. I think, however, that he has exaggerated slightly the lack of good books on physics written by mathematicians.

After a few exchanges between the speaker and various participants, the last scheduled discussion period came to an end.

Part III

RESOLUTIONS

It was the sentiment of the Organizing Committee of the Conference that, in order to bear fruit, the Conference must reach clear, well-rounded conclusions as a result of its deliberations and must make explicit recommendations for putting them into effect. This was interpreted to imply that the Conference must also propose an organized and coordinated effort for carrying out its recommendations in the various countries represented at the Conference. Accordingly, the participants, in plenary session, adopted the resolutions that are presented herewith.

THE FIRST INTER-AMERICAN CONFERENCE ON MATHEMATICAL EDUCATION

Considering

(a) That, in our technological society, mathematics is a vital branch of knowledge and an indispensable instrument for economic and social progress, particularly through its applications to biology, economics, statistics, physics, chemistry, engineering, etc.;

(b) That an alarming dearth of teachers of mathematics endangers the development of this science and of its applications;

(c) That, consequently, it is urgent to adopt measures to strengthen the training of a large number of qualified teachers, principally for the secondary school level;

(d) That the teaching of mathematics at that level must be entrusted exclusively to teachers who have received a professional training in mathematics in institutions of university rank;

(e) That, as one of the most important requirements of teaching, the teachers of mathematics must keep up to date in their field, Recommends to the Governments of the Countries of the Participants and to Authorized Agencies of These Governments

I. In connection with the training of teachers,

1. That centers for the training of high school mathematics teachers should offer scholarships and other facilities to those students who choose this career and that high school students should be informed, by means of conferences and publications, of the existence of a career as teachers and researchers in this field, and of the social importance and of the possibilities offered to those who follow it.

2. That the training of teachers of mathematics should be the sole responsibility of the university and under the influence of the most competent mathematicians, to avoid the cleavage between the teaching of mathematics and progress in science and technology. In the meantime, where this training is carried out in special institutions, mathematics courses should be of a university level.

3. That in the training of teachers of mathematics in the secondary schools, the courses should be modernized and those of a pedagogical character should be limited to proper proportions.

II. In connection with teachers in active service,

4. That regular contact be maintained between high school teachers and university professors, encouraging the former periodically to attend courses for improvement (regular or special), and that the means to achieve this end, such as scholarships at home or abroad, be increased.

5. That steps be taken to raise the socioeconomic level of the secondary school teacher holding a regular certificate, such as:

(a) Guarantee tenure.

(b) Establish basic salaries equal to those of other professions requiring similar or equivalent academic preparation.

(c) Establish a system of promotions with its corresponding implications (increase in salary, reduction of working hours, etc.) automatically based on the number of years of service, considering supplementary advantages and taking into account publications and activities aimed at self-improvement.

(d) Establish the sabbatical year.

(e) Offer the teacher the possibility of a regimen of complete dedication, as a favorable condition necessary to his progress.

6. That a maximum of incentives be assigned (scholarships, compensation, etc.) so that the teachers of the secondary school who are without certificate but are in active service can obtain one, and therefore can be covered by the system established in article 5 either by completing their university studies or by taking special courses created for this purpose.

III. In connection with the improvement of teaching,

7. That the realization of courses and the creation of institutes of an experimental character, for trying out new texts and new methods of teaching mathematics, be encouraged.

8. To suggest to the International Union of Mathematicians, UNESCO, and the Organization of American States, to take under consideration the following steps:

(a) The intensification of programs for the training of secondary school teachers of mathematics.

(b) The dispersion of activities, projects, and publications which have to do with the improvement and modernization of the teaching of mathematics.

(c) The publication and distribution of reports, new texts, and translations written for teachers of the secondary schools for their use in teaching and in self-improvement.

(d) The encouragement of research as an avenue for scientific and technological progress and as a factor in motivating teaching.

(e) The creation of an international center for the purpose of collecting and disseminating information that is relative to new experiments and new ideas in mathematics education.

(f) The creation of an Inter-American Commission on Mathematics Education, of a permanent character, for the purpose of providing continuity to the projects and ideas discussed in this Conference and to promote action calculated to raise the level and efficiency of secondary school and university teaching of mathematics.

9. To promote a wide exchange of information on new ideas in the teaching of mathematics in all countries through national meetings and other international conferences such as the present one.

10. That delegates and participants establish and maintain contact with the authorities of their respective countries, so that effective measures can be taken to put into practice these recommendations.

11. That the following persons serve as a committee pro tempore, until the establishment of a Commission on Mathematics Education in accordance with recommendation 8(f) above:

Marshall H. Stone (U.S.A.), Chairman
Bernardo Alfaro S. (Costa Rica)
Alberto González Domínguez (Argentina)
Alfredo Pereira Gómez (Brazil)
José Tola Pasquel (Peru)

VOTE OF APPRECIATION

A set of motions, drawn up by Professors Edgardo Sevilla I. (Honduras) and José Ruben Orellana (Ecuador), was presented at the closing session of the Conference and passed unanimously. The motions are as follows:

1. To approve a vote of appreciation to the Government of the Republic of Colombia and to its Minister of Education, Dr. Jaime Posada; to the Association of Colombian Universities and to its director, Professor Ulalislao Gonzalez; to the Local Organizing Committee and to its president, Dr. Pablo Casas, and its coordinator-general, Dr. Germán Zabala, for the courtesies and hospitality extended to the First Inter-American Conference on Mathematics Education.

2. To thank and congratulate the Organizing Committee of the International Commission on Mathematical Instruction; its president, Professor Marshall H. Stone; and its secretary, Professor Howard F. Fehr, for the timely initiative they took in organizing this conference.

3. To thank the following organizations and institutions for their highly esteemed contribution towards the success of the Conference:

The Organization of American States, represented by Dr. Marcelo Alonzo

The United Nations Educational, Scientific, and Cultural Organization (UNESCO), represented by Professor Oscar Dodera Luscher

The International Commission on Mathematics Instruction of the International Mathematics Union, represented by

Professor Marshall H. Stone, President of the Commission

The National Science Foundation of the United States of America, represented by Professor Bowen C. Dees

The Ford Foundation

The Rockefeller Foundation

4. To thank the distinguished mathematicians who prepared addresses on the themes in the agenda of the Conference.

5. To thank the European professors for attending the Conference and for collaborating with us:

Professor Sven Bundgaard (Denmark)
Professor Gustave Choquet (France)
Professor Laurent Pauli (Switzerland)
Professor Laurent Schwartz (France)

6. To express our gratitude to the secretarial personnel for the efficiency and patience with which they labored.

Part IV

SURVEY

BRIEF SURVEY OF MATHEMATICS EDUCATION IN THE AMERICAS

In preparing for the Conference, the Organizing Committee made an informal survey of current practices in mathematics education in the countries of the participants. The survey took the form of a questionnaire, the answers to which provided information on these topics:

1. The percentage of children in school at various age levels.
2. The extent of mathematics study at the elementary and secondary levels.
3. The qualifications and preparation of teachers; appointment procedures; salaries.
4. Opportunities afforded teachers for retraining in mathematics through special courses, publications, and professional organizations.
5. Provisions for making changes in curriculum; provision for an inspectorate.

In view of the fact that responses to the questionnaire in many instances deviated from the information sought, either because it was not available or the question was misinterpreted, it was impossible to prepare a valid comparative study. The Organizing Committee, moreover, had no intention of using the responses to contrast the educational achievements of one country with those of other countries. The results of the questionnaire are therefore presented in rather broad terms. Furthermore, no mention is made of particular practices that show a marked departure from the general program.

PERIOD OF SCHOOLING

The elementary schools are, in most instances, six-year schools, with children entering at either 6 or 7 years of age. The period of secondary schooling is generally divided into two cycles, with three years allotted to the first, and two to the second. The number of years assigned to university study is quite flexible, ranging from 4 to 7 years depending on the particular country, on the line of specialization, and on the individual student.

SCHOOL POPULATION

Elementary School

While attendance at the elementary level, in most countries, is compulsory and tuition-free, in actual practice all but two countries fail to approximate a 100 per cent attendance of the youth age 6 to 12 years. Percentages of school children in attendance in the early years of elementary school in the various countries range from 15 to 75 per cent. There are, however, great variations within a country. In many instances, for example, the rural sections offer no schooling at all. Almost all countries reported a serious dropout problem even from the first to the second year of elementary grades, with several countries stating that as many as 50 to 80 per cent of the children who start the first grade do not reach the second.

Secondary School

Countries reporting on this item were reticent about statistics on percentages of school children of secondary school age actually attending the secondary schools. Those countries submitting information indicated that 30 to 60 per cent of the group that finished elementary school entered the higher schools. The percentages of students completing elementary school, however, were reported as having been from 15 to 50 per cent of those entering, with 25 per cent the percentage quoted most frequently in this instance. All countries made mention of the fact that attendance in the secondary schools has increased tremendously in the last twenty years, with a sevenfold increase in some countries. This increase has coincided with the abolition of tuition fees for attendance in most secondary public schools.

University

The percentage of students who finish secondary school and go on to the university is very small for most countries. At the highest, it is only about 40, but in most countries, less than 5 per cent of the age group 19 to 23 years is engaged in university or other higher study.

TIME ALLOTTED TO STUDY OF MATHEMATICS

Elementary School

In the elementary school, children study mathematics for 5 periods a week in all grades. No figures are available on the amount of homework required.

Secondary School

Mathematics is a required subject in the first cycle of the secondary school, with an allotment of 5 periods per week for the first year, and 3 periods per week for each of the next two years.

During the second cycle, the student selects an area of specialization in preparation for his subsequent university study. He therefore studies mathematics only if his specialization is in the sciences or technology.

In several countries in which secondary school graduation is the only requirement for the teaching certificate in the elementary schools, there is a movement underway to make mathematics a required subject throughout the secondary school.

Several countries made note of the fact that when their students were permitted to elect their secondary school courses, the difficult subjects (mathematics, chemistry, and physics) were avoided. These countries reported that they had reverted to the prescribed program of courses.

At the secondary school level, required mathematics homework demanded an average of 1 1/2 hours per class hour.

University

Since the university years are devoted to specialization in some line or other, no mathematics is studied at this level unless the student is majoring in one of the sciences. Many countries reported that their universities had no faculty of mathematics, the subject being in the province of the engineering or the business administration faculties.

TRAINING OF TEACHERS

Teachers of the elementary schools, grades 1 through 6, are responsible for teaching all subjects. The teaching certificate is generally awarded on the basis of a secondary school diploma, and for most prospective teachers, this means that they have studied arithmetic for the six grades of the elementary school and for the three years of the first cycle of the secondary school.

Teachers of secondary school mathematics usually come from the ranks of those students who have either attended a normal school or have had some years in the university. In several countries, the law requires the secondary school teacher to be a university graduate with a major in mathematics. These laws, however, have been swept aside because of the scarcity of qualified teachers. The university student who enters the teaching profession has had his training in mathematics generally in the department of engineering or of business administration, there being, in many cases, no faculty of pure mathematics.

In recent years, it has been the practice to upgrade elementary school teachers to teach in the secondary schools, and to recruit college teachers from among the ranks of the secondary school teachers.

In only three countries is practice teaching a requirement for eligibility for a position in the secondary schools. Most normal schools fail to make provision for this on-the-job type of training. When practice teaching is provided, it is usually for a period of about 6 weeks and is generally unsupervised, with the student-teacher taking full responsibility for the class.

Appointment to teaching posts is, for most countries, in the hands of local political leaders. Promotion and salaries are generally a matter of "politics." Some countries, however, have a system of qualifying examinations for eligibility to teach. Failure on these examinations does not necessarily mean that the candidate is disqualified as a teacher. It may mean, however, that the teacher will be asked to take a less desirable position, lose promotion rights, or be required to submit to reexamination. Tenure, once achieved, is for the "life" of the person and seniority on the job is usually the main requisite for promotion.

Salaries, except in two of the countries, are so low as to rank the teaching profession at the bottom of the labor scale. Since teachers, moreover, are generally paid "by the course," it is often necessary for the teacher to work in several schools or to teach on a part-time basis while holding a job in industry or practicing some other profession.

Retraining of Teachers

Up to the present there have been only sporadic attempts to improve mathematics instruction by retraining the teacher. In the small number of countries that have provided vacation and in-service courses in contemporary mathematics, the response of teachers has not been encouraging, largely because of the lack of incentives offered and the heavy schedule of work the teachers must assume in order to eke out a living.

Perhaps the greatest obstacle to the retraining of teachers is the dearth of textbook material on the newer types of mathematics in the Spanish and Portuguese languages. There are only a few professional journals, and these do not have a wide circulation.

SELECTION OF TEXTBOOKS

In most of the Latin-American countries, it is the practice of teachers to provide the students with a list of books for the course acceptable to the teacher. The student is then free to select any one of the books as his text for the course.

EXAMINATIONS

Most countries have no form of "leaving examination" for either the elementary or the secondary schools. Even in the few instances where some form of examination is required for "graduation," the examination is a mere formality, since about 100 per cent of those taking the examination pass. Examinations are given for admission to the secondary school and to the university, but these are also rather routine and few students fail to pass.

CURRICULUM

Only five of the twenty-three countries reporting have embarked on a full-scale modernization of the mathematics curriculum in the secondary schools and universities. In no country is there a pronounced amount of "advanced" mathematics taught at the secondary school level.

Curricula, for the most part, are determined by university teachers working in cooperation with the Ministry of Education of the Central Government of the country.

INSPECTORATE

Although most countries have a central Department or Ministry of Education, the schools are autonomous in actual practice. The reason given for this situation is the scarcity of inspectors to supervise the educational programs of all the schools. Some countries, for example, could claim only three inspectors for all the secondary schools of the country.

In most countries, private schools, both religious and secular, operate under the same inspection procedures as do the public schools, and are subjected to the same curriculum restrictions.

APPENDIX

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